An explicit volume- and rank-uniform transfer-matrix gap for Wilson $SU(N)$ lattice gauge theory at strong coupling

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Abstract

We consider Wilson lattice gauge theory with gauge group $\boldsymbol{SU(N)}$, $\boldsymbol{N \geq 2}$, at fixed lattice spacing and with normalized plaquette observable $\boldsymbol{N^{-1}\,\mathrm{Re}\,\mathrm{tr}\,U_p}$. The spatial graph is finite, each plaquette uses four distinct edges, and each edge belongs to at most four plaquettes; cubic boxes and cubic tori of side at least three are included. Let $\boldsymbol{\beta_W = \mathrm{artanh}(1/14)}$ and $\boldsymbol{q(\beta) = 2\tanh\beta/(1-12\tanh\beta)}$. For $\boldsymbol{0 < \beta < \beta_W}$, we prove that the transfer operator $\boldsymbol{T_{\beta,N}}$ has a simple largest eigenvalue $\boldsymbol{\lambda_0}$ and that its remaining spectrum on the kinematical Hilbert space, hence on the gauge-invariant subspace, lies in $\boldsymbol{[0, q(\beta)\lambda_0]}$. The resulting logarithmic gap bound is independent of both spatial volume and $\boldsymbol{N}$, with the normalized-action coupling $\boldsymbol{\beta}$ held fixed. A tensor-character expansion gives strict positivity of every irreducible Fourier coefficient and an explicit trace-class identity. A time-weighted Dobrushin estimate for an auxiliary periodic measure gives temporal covariance decay. Testing this estimate on the ratio of an excited eigenfunction to the ground state and taking $\boldsymbol{n}$-th roots removes prefactors that may depend on both volume and rank. The physical Wilson transfer operator is identified by integrating temporal gauge links. No continuum statement is made.

Keywords: lattice gauge theory; Wilson action; transfer matrix; spectral gap; strong coupling; Dobrushin uniqueness

1 Introduction

We prove that for the Wilson $SU(N)$ lattice gauge theory at fixed lattice spacing and $\tanh\beta < 1/14$, one explicit number $q(\beta) < 1$ bounds the ratio of every non-leading spectral point to the leading eigenvalue of the transfer operator, simultaneously for

every integer $N \geq 2$ and every finite spatial graph in which each edge lies in at most four plaquettes. The estimate is proved directly in finite volume, on the kinematical Hilbert space and hence on the gauge-invariant one, without a thermodynamic limit, a gauge fixing or a representation cutoff.

1.1 Setting and main result

Let $G = SU(N)$, where $N \geq 2$ is any integer, with normalized Haar measure. A *spatial graph* $\Gamma = (\mathsf{V}, \mathsf{E}, \mathsf{P})$ is a finite oriented graph with a finite set P of *plaquettes*, each a closed path of four distinct edges, such that every edge lies in at most four plaquettes (Definition 2.1). The cubic box B_L with vertex set $\{0, \dots, L\}^3$ and free boundary ($L \geq 1$) and the cubic torus $\mathbb{T}_L = (\mathbb{Z}/L\mathbb{Z})^3$ ($L \geq 3$), with all unit squares as plaquettes, are spatial graphs. Let $\mathcal{H}_{\mathrm{kin}} = L^2(G^{\mathsf{E}}, \mathrm{d}U)$; the gauge group G^{V} acts by $(U^h)_e = h_{s(e)} U_e h_{t(e)}^{-1}$, and $\mathcal{H}_{\mathrm{phys}} = P_G \mathcal{H}_{\mathrm{kin}}$ is the subspace of gauge-invariant functions. With $W_p(U) = \frac{1}{N} \operatorname{Re} \operatorname{tr} U_p$ for the ordered product U_p of the link variables around p, put $\mathcal{V}_\Gamma(U) = \sum_{p \in \mathsf{P}} (1 - W_p(U))$. For $\beta > 0$ the *Wilson transfer operator* is

$$
\begin{aligned}
T_{\beta,N} = T_\beta = M_\beta K_\beta M_\beta \quad \text{on } \mathcal{H}_{\mathrm{kin}}, \qquad M_\beta = \mathrm{e}^{-\frac{\beta}{2}\mathcal{V}_\Gamma}, \\
(K_\beta f)(U) = \int_{G^{\mathsf{E}}} \prod_{e \in \mathsf{E}} w_\beta(U_e V_e^{-1})\, f(V)\, \mathrm{d}V,
\end{aligned}
\tag{1.1}
$$

with $w_\beta(u) = w_{\beta,N}(u) = \mathrm{e}^{(\beta/N) \operatorname{Re} \operatorname{tr} u} / z_{\beta,N}$ and $z_{\beta,N} = \int_G \mathrm{e}^{(\beta/N) \operatorname{Re} \operatorname{tr} v} \mathrm{d}v$. We suppress the fixed group parameter N in $M_\beta = M_{\beta,N}$, $K_\beta = K_{\beta,N}$, $T_\beta = T_{\beta,N}$, $z_\beta = z_{\beta,N}$, λ_0, and the gap notation below; L_{t} always denotes the periodic temporal extent. Proposition 2.4 shows, by integrating out the temporal links and without fixing a gauge, that $P_G T_\beta$ is the transfer operator of the isotropic Wilson measure on $\Gamma \times \mathbb{Z}_{L_{\mathrm{t}}}$ and that $P_G T_\beta = T_\beta$ on $\mathcal{H}_{\mathrm{phys}}$. Write $\lambda_0(\Gamma, \beta) = \|T_\beta\|$, Ω_Γ for the normalized ground state,

$$
\lambda_1(\Gamma, \beta) = \sup \sigma\big(T_\beta|_{\mathcal{H}_{\mathrm{phys}} \ominus \mathbb{C}\Omega_\Gamma}\big), \qquad \lambda_1^{\mathrm{kin}}(\Gamma, \beta) = \sup \sigma\big(T_\beta|_{\mathcal{H}_{\mathrm{kin}} \ominus \mathbb{C}\Omega_\Gamma}\big),
$$

$$
\Delta(\Gamma, \beta) = \log \frac{\lambda_0}{\lambda_1},
$$

the transfer-matrix gap per lattice time step (an energy gap Δ/a at lattice spacing a).

Theorem 1.1 *Let*

$$
\beta_{\mathrm{W}} := \operatorname{artanh} \tfrac{1}{14} = \tfrac{1}{2} \log \tfrac{15}{13} \quad (= 0.07155\ldots), \qquad q(\beta) := \frac{2 \tanh \beta}{1 - 12 \tanh \beta} \quad (0 < \beta < \beta_{\mathrm{W}}).
$$

Then $0 < q(\beta) < 1$ for $0 < \beta < \beta_{\mathrm{W}}$, and for every $\beta \in (0, \beta_{\mathrm{W}})$, every integer $N \geq 2$, and every spatial graph Γ in the sense of Definition 2.1 the following hold.

(i) *$\lambda_0 = \|T_{\beta,N}\|$ is a simple eigenvalue of $T_{\beta,N}$ on $\mathcal{H}_{\mathrm{kin}}$. Its eigenfunction Ω_Γ can be chosen continuous and strictly positive, and it is gauge invariant; in particular $\Omega_\Gamma \in \mathcal{H}_{\mathrm{phys}}$.*

(ii) *Every $\lambda \in \sigma(T_{\beta,N}) \setminus \{\lambda_0\}$, the spectrum being taken on $\mathcal{H}_{\mathrm{kin}}$, satisfies*

$$
0 \leq \lambda \leq q(\beta)\, \lambda_0.
$$

Consequently $\lambda_1(\Gamma, \beta) \leq \lambda_1^{\mathrm{kin}}(\Gamma, \beta) \leq q(\beta)\, \lambda_0(\Gamma, \beta)$.

(iii) *Consequently*

$$\inf_{N\geq 2,\,\Gamma}\ \Delta(\Gamma,\beta)\ \geq\ \log\frac{1}{q(\beta)}\ >\ 0,$$

the infimum being over all integers $N \geq 2$ and all spatial graphs, in particular over all boxes B_L, $L \geq 1$, and all tori $\mathbb{T}_L$, $L \geq 3$.

Quantifiers and normalization.

Fix $\beta \in (0, \beta_{\mathrm{W}})$; the integer $N \geq 2$ and graph Γ are then arbitrary. The number $q(\beta)$ depends on β alone, with no dependence on rank, volume, boundary condition, representation cutoff, or gauge choice. The action is $\beta \sum_p (1 - N^{-1} \operatorname{Re}\operatorname{tr} U_p)$, conventionally $\beta = 2N/g_0^2$. An exponent $bN \sum_p \operatorname{Re}\operatorname{tr} U_p$ instead corresponds to $\beta = N^2 b$. Thus uniformity at fixed β is not uniformity at fixed large-N parameter b or fixed 't Hooft coupling $g_0^2 N$. As $\beta \downarrow 0$, $q(\beta) = 2\beta + O(\beta^2)$.

The two Hilbert spaces.

The physical transfer operator is $P_G T_\beta$ restricted to $\mathcal{H}_{\mathrm{phys}}$ (Proposition 2.4). Theorem 1.1 is proved on the larger space $\mathcal{H}_{\mathrm{kin}}$, on which T_β is the temporal-gauge transfer matrix [1, 2], and the statement on $\mathcal{H}_{\mathrm{phys}}$ follows by restriction because P_G commutes with T_β and $\Omega_\Gamma \in \mathcal{H}_{\mathrm{phys}}$. Vectors of $\mathcal{H}_{\mathrm{kin}} \ominus \mathcal{H}_{\mathrm{phys}}$ are not physical states of the pure gauge theory; they correspond to sectors with static external charges. The proof also bounds these sectors because it does not require gauge invariance of the excited eigenfunctions.

The graph class.

The proof uses the geometry of Γ only through the number of plaquettes containing a given edge. The class therefore includes cubic boxes with free boundary and cubic tori of side at least three, and also non-cubic graphs with the same local incidence bound. The local incidence bound determines the estimate, independently of the size, shape or boundary condition of the graph.

1.2 What kind of statement this is

We distinguish finite-volume positivity, infinite-volume clustering, and a bound uniform over finite-volume transfer operators.

(a) Fixed finite volume. For each fixed Γ, the kernel of T_β is strictly positive and T_β is compact, so λ_0 is simple and $\Delta(\Gamma, \beta) > 0$ (Perron–Frobenius; Proposition 4.2). This argument provides, by itself, no lower bound on $\Delta(\Gamma, \beta)$ that is uniform in the size of Γ. Theorem 1.1 is a different statement: a single function $q(\beta)$ with $\lambda_1(\Gamma, \beta) \leq q(\beta)\lambda_0(\Gamma, \beta)$ for every admissible Γ.

(b) Infinite volume. At strong coupling, infinite-volume limits of lattice gauge theories are classical [3, 4], and exponential clustering (a mass gap in the sense of correlation decay) is attributed in the literature to the convergent strong-coupling cluster expansion (see the discussion in the introduction of [5]); exponential decay of correlations for the infinite-volume Wilson measure has also been proved by stochastic analysis [5]. Such a statement concerns one state, or the Hilbert space reconstructed from it. It

does not, without further argument, bound the spectrum of each finite-volume operator T_Γ: an eigenvector of T_Γ need not converge, as Γ grows, to a vector of a chosen limiting representation, and sectors that exist in finite volume (for instance states generated by Wilson loops winding around a torus) need not be represented there.

(c) The present statement. Theorem 1.1 controls every non-leading spectral point of every finite-volume transfer operator in the class, with the same explicit ratio, before any limit is taken. The contribution is the graph-uniform formulation, its explicit constants, and a direct finite-volume proof. We make no claim that classical expansion methods cannot yield such bounds.

1.3 The mechanism

The proof (Sections 3 and 6) uses no cluster expansion and no external theorem beyond standard functional and harmonic analysis. Schematically:

$$\begin{gathered}\text{local influence bound, uniform over all boundary data}\\ \downarrow\\ \text{time-weighted Dobrushin contraction for an auxiliary periodic measure}\\ \text{whose transfer operator is exactly } T_\beta\\ \downarrow\\ \text{temporal covariance decay at rate } \gamma \text{ in the ground state of } T_\beta\\ \downarrow\\ (\lambda/\lambda_0)^n \le C_\Gamma\, \mathrm{e}^{-\gamma n} \quad \text{via the eigenfunction ratio } \psi/\Omega_\Gamma\\ \downarrow\\ \lambda/\lambda_0 \le \mathrm{e}^{-\gamma} \quad \text{by } n\text{-th roots at fixed } N, \Gamma.\end{gathered}$$

The rate comes from the local constant $\Theta_\gamma = 1 - \tanh\beta\,(12 + 2\mathrm{e}^\gamma)$: each link has at most twelve same-time neighbours (with multiplicity) and two temporal neighbours, each with influence at most $\tanh\beta$. Optimizing over γ subject to $\Theta_\gamma > 0$ gives $q(\beta)$, and $\Theta_0 > 0$ is the condition $\beta < \beta_{\mathrm{W}}$. The prefactor $C_\Gamma = 2|\mathsf{E}|\,\|\psi/\Omega_\Gamma\|_\infty^2/\Theta_\gamma$ may depend on N, the graph, and the eigenfunction, but not on n, and $C_\Gamma^{1/n} \to 1$ at fixed N and Γ. This is the point at which all volume and rank dependence leaves the argument (Table 1).

Dobrushin's uniqueness condition [6], its block versions [7–9] and the conditional-update argument of Section 5 are standard, and we claim nothing new about them. For the transfer-operator estimate we use: the auxiliary periodic measure whose transfer operator is exactly T_β on $\mathcal{H}_{\mathrm{kin}}$ (it is not the Wilson measure; Remark 2.5); the time-weighted condition, which gives an explicit decay rate; the eigenfunction-ratio test, which converts correlation decay in the ground state into a bound on every eigenvalue; the order of limits that removes the volume-dependent prefactor; and the explicit verification of the condition for the normalized Wilson $SU(N)$ action, uniformly in N. Temporal clustering implies spectral bounds by the spectral theorem; that principle is classical. Here the eigenfunction ratio makes its use explicit for every finite-graph eigenvector.

1.4 Relation to other work

Transfer positivity and strong-coupling mass gaps are classical subjects. Creutz [1] related Wilson's lattice gauge theory [10] to the Hamiltonian formulation through the transfer matrix. Lüscher [2, Proposition 1 and its proof] constructed a positive $SU(N)$ transfer matrix and proved strict positivity of Wilson character coefficients using tensor products of the defining representation and its conjugate. We use that classical representation-theoretic argument, with the convergence and trace identity written out in Section A. Osterwalder and Seiler [3] established physical positivity, strong-coupling infinite-volume analyticity, and confinement; Seiler [4] gives a systematic constructive account. Schor [11] studied low-lying strong-coupling glueball masses. Barata and Fredenhagen [12] developed scattering theory for Euclidean lattice field theories. Neither transfer positivity nor the use of temporal correlations to constrain spectra is claimed as new here.

Shen, Zhu and Zhu [5] prove volume-uniform functional inequalities for lattice Yang–Mills Langevin dynamics, together with uniqueness and exponential correlation decay of the infinite-volume Wilson measure. Their $SU(N)$ condition is $|b| < 1/[16(d-1)]$ for the exponent $Nb\sum_p \operatorname{Re}\operatorname{tr} U_p$. Since $\beta = N^2 b$, this becomes $\beta < N^2/[16(d-1)]$ and, in $d = 4$, contains our coupling interval for every $N \geq 2$; at $N = 2$ it reads $\beta < 1/12$. Their stated correlation-decay rate depends on N, and their dynamical generator is not the Wilson transfer operator. Its Poincaré gap therefore cannot be identified directly with the transfer gap. Their correlation estimates and classical expansion methods remain relevant to transfer bounds after the required spectral conversion.

Faizal and Shabir [13, Theorem 3.3] claim a spatial-volume-uniform strong-coupling $SU(N)$ Wilson transfer gap on the Osterwalder–Schrader Hilbert space. Their positive spectral-measure argument in equations (3.45)–(3.46) is valid conditional on a common-rate, all-time correlation bound for a determining family of vectors. The preceding proof has unresolved steps: the bare Ursell bound (3.18) fails for two incompatible polymers with positive size weights, and the derivation of (B.39) bounds separation by the number of polymers rather than their total size. The finite-period setup also requires a justified ground-state limit and the one-slice density assertion used after (3.46). Thus the published argument does not establish the stated uniform gap as written. We have not verified a cited theorem supplying the missing chain with those finite-volume quantifiers. Here Propositions 2.4 to 6.6 and Lemma 6.2 give the physical identification, uniform temporal estimate, ordered limits and test of every excited eigenfunction explicitly. We do not use their theorem or their continuum claims.

Two unrefereed working notes in the navstokgap repository derive physical strong-coupling bounds on four-dimensional cubic tori with spatial sides at least four. The Dobrushin note [14, Sections 2–3], first posted in September 2026, obtains the rank-independent ratio bound $18(\mathrm{e}^{4\beta} - 1)$ and threshold $\beta < \frac{1}{4}\log(19/18) \approx 0.01352$. A companion polymer note [15, Sections 1–3] obtains the physical ratio bound $[176(\mathrm{e}^{\beta} - 1)]^4$ for $SU(3)$ at $g^2 \geq 1059$, with $\beta = 6/g^2$. Its corrected proof includes marked-observable estimates, four plaquettes per crossed time slab, decay uniform in the temporal period, and a spectral-measure argument. The physical transfer identification and periodic ground-state limit entering these conclusions are written out

explicitly here in Propositions 2.4 to 6.6 and Lemma 6.2. Both notes use the full Euclidean Wilson measure; our auxiliary measure also controls the kinematical sectors and applies to the graph class of Definition 2.1. We claim no priority for the rank-independent Dobrushin-to-gap mechanism.

The content of Theorem 1.1 is the explicit ratio with its stated graph and rank quantifiers, on the full kinematical space, and its direct finite-volume proof. We make no priority claim for strong-coupling gaps or the proof mechanism. Our search did not settle whether the same or a stronger explicit rank-uniform ratio can be extracted from the complete classical literature.

1.5 Scope

The region $\beta < \beta_{\mathrm{W}} \approx 0.0716$ is very strong coupling, and the bound $q(\beta) \approx 2\beta$ is crude. The paper contains no weak-coupling statement, no statement as $\beta \to \infty$, no continuum limit $a \to 0$, no infinite-volume statement, and no statement about a mass gap of a continuum Yang–Mills theory. All integers $N \geq 2$ are covered, with the normalized-action coupling held fixed.

Section 2 defines the model and proves the identification with the Wilson measure; Section 3 gives the roadmap and the volume bookkeeping; Sections 4 to 6 contain the proof; Section 7 discusses limitations.

2 Wilson $SU(N)$ lattice gauge theory and its transfer operator

2.1 Spatial graphs, configurations and the gauge group

Definition 2.1 (Spatial graph) A *spatial graph* is a triple $\Gamma = (\mathsf{V}, \mathsf{E}, \mathsf{P})$, where (V, E) is a finite oriented graph and P is a finite set of *plaquettes*. Each plaquette is a closed path $e_1^{\sigma_1} e_2^{\sigma_2} e_3^{\sigma_3} e_4^{\sigma_4}$ ($\sigma_i = \pm 1$) of four distinct edges. Every edge lies in at most four plaquettes. An edge e runs from $s(e)$ to $t(e)$.

The cubic box B_L ($L \geq 1$) has vertex set $\{0, \dots, L\}^3$, nearest-neighbour edges oriented in the positive coordinate directions, and all unit squares as plaquettes. The cubic torus $\mathbb{T}_L$ ($L \geq 3$) has vertex set $(\mathbb{Z}/L\mathbb{Z})^3$, $3L^3$ edges and $3L^3$ unit squares. In both cases every edge lies in at most four plaquettes, and the four edges of a unit square are distinct; we restrict to $L \geq 3$ on the torus for definiteness. No boundary condition is imposed on links and no gauge is fixed. The proofs below use only the properties listed in Definition 2.1.

Let $G = SU(N)$, where $N \geq 2$ is any integer, with normalized Haar measure $\mathrm{d}u$. A *configuration* is $U = (U_e)_{e \in \mathsf{E}} \in G^{\mathsf{E}}$, with product Haar measure $\mathrm{d}U$. For a plaquette $p = e_1^{\sigma_1} \cdots e_4^{\sigma_4}$ put $U_p = U_{e_1}^{\sigma_1} \cdots U_{e_4}^{\sigma_4}$ and

$$W_p(U) = \tfrac{1}{N} \operatorname{Re} \operatorname{tr} U_p \in [-1, 1], \qquad \mathcal{V}_\Gamma(U) = \sum_{p \in \mathsf{P}} \bigl(1 - W_p(U)\bigr) \in [0, 2|\mathsf{P}|].$$

Since $\operatorname{Re}\operatorname{tr} g^{-1} = \operatorname{Re}\operatorname{tr} g$ on $SU(N)$, W_p depends neither on the starting point nor on the orientation of the closed path. The gauge group $\mathcal{G} = G^{\mathsf{V}}$ acts on configurations by $(U^h)_e = h_{s(e)} U_e h_{t(e)}^{-1}$, and $W_p(U^h) = W_p(U)$.

Definition 2.2 (Hilbert spaces) $\mathcal{H}_{\mathrm{kin}} = L^2(G^{\mathsf{E}}, \mathrm{d}U)$. For $h \in \mathcal{G}$ let $(\mathcal{R}_h f)(U) = f(U^h)$, and let $P_G = \int_{\mathcal{G}} \mathcal{R}_h \, \mathrm{d}h$ be the average over the compact group $\mathcal{G}$ with its normalized Haar measure. Then P_G is the orthogonal projection onto the gauge-invariant subspace $\mathcal{H}_{\mathrm{phys}} = P_G \mathcal{H}_{\mathrm{kin}} = \{f : \mathcal{R}_h f = f \ \forall h\}$.

2.2 The transfer operator

For $\beta > 0$ let

$$w_\beta(u) = \frac{\mathrm{e}^{\frac{\beta}{N} \operatorname{Re}\operatorname{tr} u}}{z_\beta}, \qquad z_\beta = \int_G \mathrm{e}^{\frac{\beta}{N} \operatorname{Re}\operatorname{tr} v} \, \mathrm{d}v. \tag{2.1}$$

This is a continuous, strictly positive probability density on G, central ($w_\beta(vuv^{-1}) = w_\beta(u)$) and inversion invariant ($w_\beta(u^{-1}) = w_\beta(u)$, since $\operatorname{Re}\operatorname{tr} u^{-1} = \operatorname{Re}\operatorname{tr} u$). For a function k on G write $(k * f)(u) = \int_G k(uv^{-1}) f(v) \, \mathrm{d}v$.

Definition 2.3 (Wilson transfer operator) On $\mathcal{H}_{\mathrm{kin}}$ let M_β be multiplication by $\mathrm{e}^{-\frac{\beta}{2}\mathcal{V}_\Gamma}$, let $K_\beta = \bigotimes_{e \in \mathsf{E}} (w_\beta * \cdot)_e$, and let

$$T_\beta = M_\beta K_\beta M_\beta.$$

Its integral kernel is

$$T_\beta(U, V) = \mathrm{e}^{-\frac{\beta}{2}\mathcal{V}_\Gamma(U)} \Big(\prod_{e \in \mathsf{E}} w_\beta(U_e V_e^{-1}) \Big) \mathrm{e}^{-\frac{\beta}{2}\mathcal{V}_\Gamma(V)}. \tag{2.2}$$

M_β and K_β commute with every $\mathcal{R}_h$: for M_β this is gauge invariance of $\mathcal{V}_\Gamma$; for K_β, $\mathcal{R}_h$ acts on each U_e by a left and a right translation, which commute with convolution by a central function. Hence T_β commutes with P_G, and $\mathcal{H}_{\mathrm{phys}}$ and $\mathcal{H}_{\mathrm{kin}} \ominus \mathcal{H}_{\mathrm{phys}}$ are invariant under T_β.

2.3 Identification with the Wilson measure

Let $L_{\mathrm{t}} \geq 2$. The Wilson lattice gauge theory on $\Gamma \times \mathbb{Z}_{L_{\mathrm{t}}}$ has variables $U_{e,\tau}$ on the spatial links ($e \in \mathsf{E}$, $\tau \in \mathbb{Z}_{L_{\mathrm{t}}}$) and $h_{v,\tau}$ on the temporal links from (v, τ) to $(v, \tau + 1)$. Its spatial plaquettes are the pairs (p, τ); its temporal plaquettes are the pairs (e, τ), with plaquette variable

$$Y_{e,\tau} = U_{e,\tau} \, h_{t(e),\tau} \, U_{e,\tau+1}^{-1} \, h_{s(e),\tau}^{-1}.$$

The isotropic Wilson action $\beta \sum_{\text{plaquettes}} (1 - \frac{1}{N} \operatorname{Re}\operatorname{tr} U_{\text{plaquette}})$ defines the probability measure

$$\mathrm{d}\mu_{L_{\mathrm{t}}}^{\mathrm{W}} = \frac{1}{Z_{L_{\mathrm{t}}}^{\mathrm{W}}} \prod_{\tau \in \mathbb{Z}_{L_{\mathrm{t}}}} \mathrm{e}^{-\beta \mathcal{V}_\Gamma(U_\tau)} \prod_{\tau \in \mathbb{Z}_{L_{\mathrm{t}}}} \prod_{e \in \mathsf{E}} \mathrm{e}^{\frac{\beta}{N} \operatorname{Re}\operatorname{tr} Y_{e,\tau}} \prod_{e,\tau} \mathrm{d}U_{e,\tau} \prod_{v,\tau} \mathrm{d}h_{v,\tau}, \tag{2.3}$$

where $U_\tau = (U_{e,\tau})_{e\in\mathsf{E}}$ and the constants $\mathrm{e}^{-\beta}$ of the temporal plaquettes have been absorbed into $Z^{\mathrm{W}}_{L_\mathrm{t}}$. This measure is invariant under all gauge transformations of $\Gamma \times \mathbb{Z}_{L_\mathrm{t}}$; no gauge is fixed.

Proposition 2.4 *For $L_\mathrm{t} \geq 2$ and bounded measurable functions $f_0, \dots, f_{L_\mathrm{t}-1}$ on G^{E},*

$$\int \prod_{\tau\in\mathbb{Z}_{L_\mathrm{t}}} f_\tau(U_\tau)\,\mathrm{d}\mu^{\mathrm{W}}_{L_\mathrm{t}} = \frac{\mathrm{tr}_{\mathcal{H}_{\mathrm{kin}}}\big(f_0 P_G T_\beta\, f_1 P_G T_\beta \cdots f_{L_\mathrm{t}-1} P_G T_\beta\big)}{\mathrm{tr}_{\mathcal{H}_{\mathrm{kin}}}\big((P_G T_\beta)^{L_\mathrm{t}}\big)},$$

where f_τ also denotes the multiplication operator. Thus $P_G T_\beta$ is the transfer operator of $\mu^{\mathrm{W}}_{L_\mathrm{t}}$, and on $\mathcal{H}_{\mathrm{phys}}$ it coincides with T_β.

Proof Integration of the temporal links. For fixed spatial variables, the temporal variables $(h_{v,\tau})_{v\in\mathsf{V}}$ of a given slab τ occur only in the factors $\mathrm{e}^{\frac{\beta}{N}\mathrm{Re}\,\mathrm{tr}\,Y_{e,\tau}} = z_\beta\, w_\beta(Y_{e,\tau})$, $e \in \mathsf{E}$. Since w_β is central, $w_\beta(Y_{e,\tau}) = w_\beta\big(g_{s(e)}U_{e,\tau}g^{-1}_{t(e)}U^{-1}_{e,\tau+1}\big)$ with $g_v = h^{-1}_{v,\tau}$, i.e. $w_\beta(Y_{e,\tau}) = w_\beta\big((U^g_\tau)_e U^{-1}_{e,\tau+1}\big)$. Since Haar measure on $\mathcal{G}$ is inversion invariant,

$$\int \prod_{e\in\mathsf{E}} w_\beta(Y_{e,\tau}) \prod_{v\in\mathsf{V}} \mathrm{d}h_{v,\tau} = \int_{\mathcal{G}} K_\beta\big(U^g_\tau, U_{\tau+1}\big)\,\mathrm{d}g =: (P_G K_\beta)(U_\tau, U_{\tau+1}),$$

where $K_\beta(U,V) = \prod_e w_\beta(U_e V_e^{-1})$. This is the integral kernel of $P_G K_\beta$, because $(P_G K_\beta f)(U) = \int_{\mathcal{G}} (K_\beta f)(U^g)\,\mathrm{d}g$. It is bounded and continuous.

Trace formula. Moreover $\mathrm{e}^{-\beta\mathcal{V}_\Gamma(U_\tau)} = M_\beta(U_\tau)^2$. After integrating all temporal links, Fubini's theorem gives, up to a positive constant that cancels on normalization,

$$\begin{aligned}\int \prod_\tau f_\tau(U_\tau)\,\mathrm{d}\mu^{\mathrm{W}}_{L_\mathrm{t}} &\propto \int \prod_{\tau\in\mathbb{Z}_{L_\mathrm{t}}} M_\beta(U_\tau)^2 f_\tau(U_\tau)\,(P_G K_\beta)(U_\tau, U_{\tau+1}) \prod_\tau \mathrm{d}U_\tau \\ &= \mathrm{tr} \prod_{\tau=0}^{L_\mathrm{t}-1} \big(M^2_\beta f_\tau P_G K_\beta\big),\end{aligned}$$

by Lemma B.1, with the product ordered by increasing τ and $U_{L_\mathrm{t}} := U_0$.

Conjugation. M_β commutes with f_τ and with P_G, so $M^2_\beta f_\tau P_G K_\beta = M_\beta\,(f_\tau P_G T_\beta)\,M^{-1}_\beta$, where M_β is bounded with bounded inverse. The product telescopes to $M_\beta\big(\prod_\tau f_\tau P_G T_\beta\big)M^{-1}_\beta$, whose trace is $\mathrm{tr}\prod_\tau (f_\tau P_G T_\beta)$. Normalizing by the case $f_\tau \equiv 1$ gives the formula. Finally, $P_G T_\beta = T_\beta P_G$, so $P_G T_\beta = T_\beta$ on $\mathcal{H}_{\mathrm{phys}}$. □

Restoring the constants omitted from the normalized density multiplies each slab operator by $(\mathrm{e}^{-\beta} z_\beta)^{|\mathsf{E}|}$. This scalar leaves every spectral ratio unchanged. The construction uses only oriented paths and Haar invariance, so it does not require connectedness or trivial spatial topology.

By Lemma 4.1 and Proposition 4.2, $T_\beta \geq 0$ and $P_G T_\beta = P_G T_\beta P_G \geq 0$. This supplies the positive physical one-step transfer without assuming that a pointwise positive kernel is a positive operator. It also gives temporal reflection positivity for the integrated spatial-slice measure. On an even circle of length $2m$, put $A = P_G T_\beta$ and, for a product of half-chain observables, $B = A^{1/2} f_0 A \cdots A f_{m-1} A^{1/2}$. Reflection reverses the order and takes adjoints, and the cyclic trace formula gives the reflected expectation as $\mathrm{tr}(B^* B)/\mathrm{tr}(A^{2m}) \geq 0$. Finite linear combinations give the same formula

with B replaced by a sum. The operators are Hilbert–Schmidt since $A^{1/2}$ is; uniform approximation extends the inequality to continuous half-chain observables. This is the temporal positivity relevant here; no continuum reconstruction is invoked.

Thus the gap of T_β on $\mathcal{H}_{\mathrm{phys}}$ is the gap of the transfer operator of the standard Wilson theory, defined without gauge fixing. The larger space $\mathcal{H}_{\mathrm{kin}}$ is the space of the temporal-gauge transfer matrix [1, 2]; its non-invariant vectors describe static external charges. Theorem 1.1 is proved on $\mathcal{H}_{\mathrm{kin}}$, which is a stronger statement.

Remark 2.5 (Two different measures) The proof in Section 6 uses a second measure $\tilde{\mu}_{L_t}$ (Definition 6.1), obtained from (2.3) by setting every temporal link variable equal to 1. It is *not* the Wilson measure in a gauge: on the periodic lattice $\Gamma \times \mathbb{Z}_{L_t}$ the temporal links cannot all be gauged to 1, and integrating the temporal links produces P_G in Proposition 2.4. The measure $\tilde{\mu}_{L_t}$ is a spin system whose transfer operator is T_β on all of $\mathcal{H}_{\mathrm{kin}}$ (Lemma 6.2). Its role is purely technical. The physical identification of T_β rests on Proposition 2.4 alone, and no property of $\tilde{\mu}_{L_t}$ is attributed to the Wilson measure.

If $\mathsf{E} = \varnothing$, then $\mathsf{P} = \varnothing$, G^{E} is a point, $\mathcal{H}_{\mathrm{kin}} = \mathcal{H}_{\mathrm{phys}} = \mathbb{C}$, and $T_\beta = 1$. The theorem holds with no excited spectrum and gap $+\infty$. The mixing and spectral-conversion arguments below are applied only when $\mathsf{E} \neq \varnothing$.

Definition 2.6 (Transfer gap) Let $\lambda_0 = \lambda_0(\Gamma, \beta) = \|T_\beta\|$ and let Ω_Γ be the normalized eigenvector of Proposition 4.2(iii). Put

$$\lambda_1(\Gamma, \beta) = \sup \sigma\big(T_\beta|_{\mathcal{H}_{\mathrm{phys}} \ominus \mathbb{C}\Omega_\Gamma}\big), \qquad \lambda_1^{\mathrm{kin}}(\Gamma, \beta) = \sup \sigma\big(T_\beta|_{\mathcal{H}_{\mathrm{kin}} \ominus \mathbb{C}\Omega_\Gamma}\big),$$

with the convention $\sup \emptyset = 0$, and $\Delta(\Gamma, \beta) = \log(\lambda_0/\lambda_1) \in (0, \infty]$.

If $\mathsf{P} \neq \emptyset$, then $\mathcal{H}_{\mathrm{phys}}$ is infinite dimensional: for a plaquette p the functions $U \mapsto \chi_\pi(U_p)$, $\pi \in \widehat{SU(N)}$, are gauge invariant and orthonormal, because U_p is Haar distributed. Then $\lambda_1(\Gamma, \beta) > 0$ by injectivity of T_β (Proposition 4.2), and $\Delta(\Gamma, \beta) < \infty$.

3 Strategy of the proof and the origin of volume and rank uniformity

Fix an integer $N \geq 2$, $\beta \in (0, \beta_{\mathrm{W}})$, and a spatial graph Γ, and write $T = T_\beta$, $\lambda_0 = \|T\|$, $\Omega = \Omega_\Gamma$, $S = T/\lambda_0$. By Proposition 4.2, T is compact, self-adjoint and positive on $\mathcal{H}_{\mathrm{kin}}$, with a simple top eigenvalue and a continuous eigenfunction Ω satisfying $\min \Omega > 0$, and every eigenfunction of T is continuous. Let $\lambda \neq \lambda_0$ be an eigenvalue with a real normalized eigenfunction $\psi \perp \Omega$, and put $b = \psi/\Omega$. Then b is a bounded continuous real function, $b\,\Omega = \psi$, $\langle \Omega, b\,\Omega \rangle = 0$, and

$$\Big(\frac{\lambda}{\lambda_0}\Big)^n = \langle b\,\Omega, S^n\, b\,\Omega \rangle - \langle \Omega, b\,\Omega \rangle^2 \qquad (n \geq 0). \tag{3.1}$$

The right-hand side is a time correlation of the multiplication observable b in the ground state of T. The proof bounds it in three steps.

Step 1: a measure with transfer operator T. On $(G^{\mathsf{E}})^{\mathbb{Z}_{L_\mathrm{t}}}$ let $\tilde{\mu}_{L_\mathrm{t}}$ have density proportional to $\prod_\tau \mathrm{e}^{-\beta \mathcal{V}_\Gamma(U_\tau)} \prod_{\tau,e} w_\beta(U_{e,\tau} U_{e,\tau+1}^{-1})$. By a trace formula (Lemma 6.2), its two-time correlations converge, as $L_\mathrm{t} \to \infty$ at fixed n, to the right-hand side of (3.1).

Step 2: uniform mixing. The conditional law of one link variable of $\tilde{\mu}_{L_\mathrm{t}}$ given all others depends on the value of another link by at most $\tanh\beta$ per shared plaquette or temporal bond, in total variation, uniformly over the values of all remaining links (Lemma 6.3). Each link has at most twelve same-time neighbours (with multiplicity) and two temporal neighbours. With the weights $\mathrm{e}^{\gamma\cdot(\text{time distance to } n)}$, the weighted Dobrushin constant is therefore at least $\Theta_\gamma = 1 - \tanh\beta\,(12 + 2\mathrm{e}^{\gamma})$, independently of N, Γ, L_t and n (Lemma 6.4). The general estimate of Section 5 turns this into

$$\left|\mathrm{Cov}_{\tilde{\mu}_{L_\mathrm{t}}}\big(b(U_0), b(U_n)\big)\right| \le \frac{2|\mathsf{E}|\,\|b\|_\infty^2}{\Theta_\gamma}\,\mathrm{e}^{-\gamma n} \qquad \text{for all } L_\mathrm{t} \ge \max(3, 2n).$$

Step 3: n-th roots. Letting $L_\mathrm{t} \to \infty$ and inserting (3.1) gives $(\lambda/\lambda_0)^n \le C_\Gamma \mathrm{e}^{-\gamma n}$ with $C_\Gamma = 2|\mathsf{E}|\,\|\psi/\Omega\|_\infty^2/\Theta_\gamma$, for every $n \ge 1$. Taking n-th roots at fixed N and Γ and letting $n \to \infty$ yields $\lambda/\lambda_0 \le \mathrm{e}^{-\gamma}$. Letting e^{γ} increase to $(1 - 12\tanh\beta)/(2\tanh\beta)$, the supremum allowed by $\Theta_\gamma > 0$, gives $\lambda/\lambda_0 \le q(\beta)$.

Order of limits.

The rank parameter N and graph Γ are fixed throughout. For each fixed n, the covariance bound holds uniformly in $L_\mathrm{t} \ge \max(3, 2n)$, and $L_\mathrm{t} \to \infty$ is taken at fixed n, N, and Γ. This gives $(\lambda/\lambda_0)^n \le C_\Gamma \mathrm{e}^{-\gamma n}$ for every n. Only then is $n \to \infty$ taken, after n-th roots, giving $\lambda/\lambda_0 \le \mathrm{e}^{-\gamma}$. Finally γ is optimized. No rank or spatial-volume limit is taken anywhere, and no estimate is interchanged with such a limit.

Where rank and volume enter.

Table 1 lists the potentially volume- or rank-dependent quantities in the proof.

4 Perron–Frobenius and single-link structure

Lemma 4.1 (Single-link operator) *Fix $N \ge 2$ and $\beta > 0$. Let F be the defining representation of $G = SU(N)$, and write*

$$(F \oplus \overline{F})^{\otimes k} = \bigoplus_{\pi \in \widehat{G}} m_{k,\pi}\,\pi, \qquad c_\pi = \sum_{k \ge 0} \frac{(\beta/(2N))^k}{k!}\, m_{k,\pi}.$$

Every c_π is strictly positive. The convolution operator $w_\beta * \cdot$ acts on the d_π^2-dimensional matrix-coefficient space of π as multiplication by

$$\kappa_\pi = \frac{c_\pi}{d_\pi z_\beta} > 0, \qquad \kappa_{\mathbf{1}} = 1, \qquad \sum_{\pi \in \widehat{G}} d_\pi^2 \kappa_\pi = \frac{\mathrm{e}^{\beta}}{z_\beta} < \infty.$$

It is a positive, injective, self-adjoint, trace-class contraction. The finite tensor product K_β has the same properties.

Quantity	Where it enters	Why it does not enter the rate
$\lvert\mathsf{E}\rvert$	Prefactor of the covariance bound (Proposition 6.5), from summing $\delta_i(f)$ over the sites of one time slice	Part of C_Γ, which is independent of n; removed by $C_\Gamma^{1/n} \to 1$ (Proposition 6.6)
$\lVert\psi/\Omega\rVert_\infty$, hence $\min\Omega$	$\lVert b\rVert_\infty$ in the prefactor (Proposition 6.6); not controlled uniformly in N or Γ by our argument	Same: part of C_Γ, removed by the n-th root at fixed N and Γ
λ_0, $\operatorname{tr} T$, the second eigenvalue of T	Speed of convergence in the limit $L_\mathrm{t} \to \infty$ (Lemma 6.2)	The limit is qualitative, taken at fixed n, N, and Γ; the bound it is applied to is uniform in L_t
$\lvert\mathsf{P}\rvert$	Bounds $\mathrm{e}^{-2\beta\lvert\mathsf{P}\rvert} \le M_\beta^2 \le 1$, used for boundedness and invertibility of M_β (Proposition 4.2)	Used only qualitatively
$\lvert I_{L_\mathrm{t}}\rvert = L_\mathrm{t}\lvert\mathsf{E}\rvert$	Number of sites in the Glauber dynamics (Proposition 5.3)	Cancels exactly in the contraction rate c_w
Rate $\mathrm{e}^{-\gamma}$	$\Theta_\gamma = 1 - \tanh\beta(12 + 2\mathrm{e}^\gamma)$ (Lemma 6.4)	Uses only: ≤ 4 plaquettes per edge, 3 other edges per plaquette, 2 temporal neighbours, $\lvert N^{-1}\operatorname{Re}\operatorname{tr} g\rvert \le 1$

Table 1 Volume- or rank-dependent quantities in the proof of Theorem 1.1.

Proof Lemma A.1 proves the absolutely and uniformly convergent expansion $\mathrm{e}^{(\beta/N)\operatorname{Re}\operatorname{tr} g} = \sum_\pi c_\pi \chi_\pi(g)$, with $c_\pi > 0$, $z_\beta = c_{\mathbf{1}}$, and $\sum_\pi d_\pi c_\pi = \mathrm{e}^\beta$. Schur orthogonality gives the scalar $c_\pi/(d_\pi z_\beta)$ on each matrix-coefficient space. Peter–Weyl gives a complete orthogonal decomposition into these spaces, proving positivity, self-adjointness and injectivity. Young's inequality gives $\lVert w_\beta * f\rVert_2 \le \lVert w_\beta\rVert_1 \lVert f\rVert_2 = \lVert f\rVert_2$. The displayed sum is finite, proving trace class, and finite tensor products retain the assertions. Positivity here follows from the Fourier coefficients, not merely from the pointwise positivity of w_β. □

Proposition 4.2 (Perron–Frobenius structure) *Let $\beta > 0$ and let Γ be a spatial graph. Then:*

(i) *T_β is a positive, injective, trace-class, self-adjoint contraction on $\mathcal{H}_\mathrm{kin}$, and it commutes with P_G.*

(ii) *T_β has the continuous, strictly positive, real symmetric integral kernel (2.2). It maps $\mathcal{H}_\mathrm{kin}$ into continuous functions, and every eigenfunction of T_β is continuous.*

(iii) *The largest eigenvalue $\lambda_0 = \lVert T_\beta\rVert$ is simple. Its eigenspace is spanned by a continuous function Ω_Γ with $\min\Omega_\Gamma > 0$, which is gauge invariant. In particular $\Omega_\Gamma \in \mathcal{H}_\mathrm{phys}$.*

(iv) *$\sigma(T_\beta) \setminus \{0\}$ consists of eigenvalues of finite multiplicity in $(0, \lambda_0]$. Every eigenvalue $\lambda \ne \lambda_0$ has a real eigenfunction orthogonal to Ω_Γ.*

Proof Write $T = T_\beta$, $M = M_\beta$, $K = K_\beta$.

(i) By Lemma 4.1, K is a positive, injective, trace-class contraction. M is multiplication by a continuous function with values in $[\mathrm{e}^{-\beta|\mathsf{P}|}, 1]$, hence a self-adjoint contraction with bounded inverse. Therefore $T = (K^{1/2}M)^*(K^{1/2}M) \geq 0$ is trace class with $\|T\| \leq 1$, and $Tf = 0$ implies $\|K^{1/2}Mf\|^2 = \langle f, Tf\rangle = 0$, hence $Mf = 0$ by injectivity of $K^{1/2}$, hence $f = 0$. Commutation with P_G was shown after Definition 2.3.

(ii) The kernel (2.2) is continuous and strictly positive because w_β and $\mathrm{e}^{-\frac{\beta}{2}\mathcal{V}_\Gamma}$ are. It is real and symmetric because $w_\beta(u^{-1}) = w_\beta(u)$. For $f \in \mathcal{H}_{\mathrm{kin}}$, $Tf(U) = \int T(U,V)f(V)\,\mathrm{d}V$ is continuous by uniform continuity of the kernel on the compact space $G^{\mathsf{E}} \times G^{\mathsf{E}}$ and the Cauchy–Schwarz inequality. If $Tf = \lambda f$ with $\lambda \neq 0$, then $f = \lambda^{-1}Tf$ is continuous; 0 is not an eigenvalue by (i).

(iii) Since $T \geq 0$ is compact and nonzero, $\lambda_0 = \|T\| = \max \sigma(T)$ is an eigenvalue. Let ψ be an eigenvector. Since the kernel is real, the real and imaginary parts of ψ are eigenvectors, so we may take ψ real. Since the kernel is positive, $\langle|\psi|, T|\psi|\rangle \geq \langle\psi, T\psi\rangle = \lambda_0\|\psi\|^2$, so $|\psi|$ maximizes the quadratic form and is also an eigenvector for λ_0. Then $|\psi| = \lambda_0^{-1}T|\psi|$ is the integral of a strictly positive continuous kernel against a nonnegative nonzero function, hence continuous and strictly positive, and $\min|\psi| > 0$ by compactness.

Suppose the eigenspace had dimension at least two. Then it would contain a real $\varphi \neq 0$ with $\varphi \perp |\psi|$; since $|\psi| > 0$, such a φ takes both signs on sets of positive measure. Writing $\varphi = \varphi_+ - \varphi_-$,

$$\langle|\varphi|, T|\varphi|\rangle - \langle\varphi, T\varphi\rangle = 4\langle\varphi_+, T\varphi_-\rangle > 0,$$

because the kernel is strictly positive and $\varphi_\pm \neq 0$. This contradicts $\langle|\varphi|, T|\varphi|\rangle \leq \lambda_0\|\varphi\|^2 = \langle\varphi, T\varphi\rangle$. So the eigenspace is spanned by $\Omega_\Gamma := |\psi|/\||\psi|\|$. For $h \in \mathcal{G}$, $\mathcal{R}_h\Omega_\Gamma$ is again a positive normalized eigenvector for λ_0, because $\mathcal{R}_h$ is unitary, preserves positivity and commutes with T; hence $\mathcal{R}_h\Omega_\Gamma = \Omega_\Gamma$.

(iv) T is compact and self-adjoint, so $\sigma(T)\backslash\{0\}$ consists of eigenvalues of finite multiplicity, which lie in $(0, \lambda_0]$ by (i). Eigenvectors of distinct eigenvalues are orthogonal, and since the kernel is real, real and imaginary parts of eigenfunctions are eigenfunctions. □

Remark 4.3 Proposition 4.2 holds for every $\beta > 0$ and every spatial graph. It gives $\lambda_1^{\mathrm{kin}}(\Gamma, \beta) < \lambda_0(\Gamma, \beta)$ on each graph, but by itself it gives no lower bound on the ratio $\lambda_0/\lambda_1^{\mathrm{kin}}$ that is uniform in Γ. It is used below only qualitatively: for the existence of a strictly positive continuous ground state, and for the continuity of excited eigenfunctions.

5 A weighted single-site Dobrushin estimate

This section is independent of gauge theory. The argument is the classical one behind Dobrushin's uniqueness condition [6], in the form used for block conditions by Dobrushin and Shlosman [7] and Weitz [8]. We give the single-site version with an arbitrary weight, because the time weights are what produce the decay rate, and because we need the conditional laws defined at every point rather than almost everywhere.

5.1 Spin systems and conditional laws

All functions used for tilts and oscillations in this section are real. A *spin system* consists of a nonempty finite set I; for each $i \in I$ a compact metrizable space X_i with a Borel probability measure m_i; and a bounded measurable function H on $X = \prod_i X_i$.

Its Gibbs measure is $\mu(\mathrm{d}\sigma) = Z^{-1}\mathrm{e}^{-H(\sigma)}m(\mathrm{d}\sigma)$, with $m = \bigotimes_i m_i$. For $i \in I$ and $\sigma \in X$ write $\sigma = (\sigma_i, \sigma_{i^c})$. The *conditional law* of the site i given σ is the probability measure on X_i

$$\mu_i^\sigma(\mathrm{d}\eta) = \frac{\mathrm{e}^{-H(\eta\,\sigma_{i^c})}}{\int \mathrm{e}^{-H(\eta'\sigma_{i^c})}m_i(\mathrm{d}\eta')}\, m_i(\mathrm{d}\eta),$$

defined for every σ and depending only on σ_{i^c}. The *single-site update* acts on bounded measurable f by

$$(Q_i f)(\sigma) = \int f(\eta\,\sigma_{i^c})\,\mu_i^\sigma(\mathrm{d}\eta).$$

By Fubini's theorem:

(Q1) $Q_i f$ is bounded measurable, Q_i is positive, $Q_i 1 = 1$, and $Q_i f$ does not depend on σ_i;
(Q2) $Q_i(u\,Q_i v) = Q_i u \cdot Q_i v$;
(Q3) $\mu(Q_i f) = \mu(f)$.

For $j \in I$ write $\sigma \sim_j \sigma'$ if $\sigma_k = \sigma'_k$ for all $k \neq j$. For bounded f let

$$\delta_j(f) = \sup_{\sigma \sim_j \sigma'} |f(\sigma) - f(\sigma')| \le 2\|f\|_\infty.$$

Changing one coordinate at a time gives $|f(\sigma) - f(\sigma')| \le \sum_{j:\sigma_j \neq \sigma'_j} \delta_j(f)$.

The *total variation distance* of probability measures ν, ν' is $\mathrm{TV}(\nu,\nu') = \sup_A |\nu(A) - \nu'(A)|$. For bounded measurable φ,

$$|\nu(\varphi) - \nu'(\varphi)| \le \mathrm{osc}(\varphi)\ \mathrm{TV}(\nu,\nu'), \qquad \mathrm{osc}(\varphi) = \sup\varphi - \inf\varphi. \tag{5.1}$$

This holds because $\nu - \nu'$ has total mass zero: replace φ by $\varphi - \inf\varphi$, and use that the positive and negative parts of $\nu - \nu'$ each have mass $\mathrm{TV}(\nu,\nu')$.

Definition 5.1 (Influence and weighted Dobrushin constant) For $i \neq j$ the *influence* of j on i is

$$C_{ij} = \sup_{\sigma \sim_j \sigma'} \mathrm{TV}\big(\mu_i^\sigma, \mu_i^{\sigma'}\big) \in [0,1].$$

A *weight* is a function $w : I \to (0,\infty)$. The *weighted Dobrushin constant* is

$$c_w = \min_{j \in I}\Big(1 - \sum_{i \neq j} \frac{w_j}{w_i}\, C_{ij}\Big),$$

and we put $\Delta_w(f) = \max_j w_j\,\delta_j(f)$.

The supremum in C_{ij} is over all configurations, not only typical ones. Note the orientation: C_{ij} measures the influence of the site j on the conditional law at i, and c_w is a minimum over j of weighted sums over i (column sums of the influence matrix). This is the orientation produced by Lemma 5.2.

5.2 Contraction and covariance

Lemma 5.2 *For bounded measurable f, $\delta_i(Q_i f) = 0$, and for $j \neq i$,*

$$\delta_j(Q_i f) \le \delta_j(f) + C_{ij}\,\delta_i(f) \le \delta_j(f) + \frac{C_{ij}}{w_i}\Delta_w(f).$$

Proof The first statement is (Q1). Let $j \neq i$ and $\sigma \sim_j \sigma'$. Then

$$Q_i f(\sigma) - Q_i f(\sigma') = \int \big[f(\eta\,\sigma_{i^c}) - f(\eta\,\sigma'_{i^c})\big]\mu_i^{\sigma}(\mathrm{d}\eta) + \big[\mu_i^{\sigma}(\varphi) - \mu_i^{\sigma'}(\varphi)\big], \qquad \varphi(\eta) = f(\eta\,\sigma'_{i^c}).$$

In the first term the two configurations differ only at j, so it is at most $\delta_j(f)$ in absolute value. The function φ depends only on the coordinate i, so $\operatorname{osc}\varphi \le \delta_i(f)$, and (5.1) bounds the second term by $\delta_i(f)C_{ij}$. Finally $\delta_i(f) \le \Delta_w(f)/w_i$. □

Proposition 5.3 (Contraction) *Let $L = \sum_{i\in I}(Q_i - 1)$ and $P_s = \mathrm{e}^{sL}$ $(s \ge 0)$, acting on bounded measurable functions. Then P_s is positive, $P_s 1 = 1$, $\mu P_s = \mu$, and*

$$\Delta_w(P_s f) \le \mathrm{e}^{-c_w s}\,\Delta_w(f) \qquad (s \ge 0).$$

If $c_w > 0$, then $\|P_s f - \mu(f)\|_\infty \le \mathrm{e}^{-c_w s}\Delta_w(f)\sum_j w_j^{-1} \to 0$ as $s \to \infty$.

Proof L is bounded in the supremum norm, and $P_s = \mathrm{e}^{-|I|s}\sum_{k\ge 0}\frac{s^k}{k!}\big(\sum_i Q_i\big)^k$. This gives positivity and $P_s 1 = 1$; together with (Q3) it gives $\mu P_s = \mu$. By Lemma 5.2 and subadditivity of δ_j,

$$\begin{aligned} w_j\,\delta_j\Big(\sum_i Q_i f\Big) &\le \sum_{i\neq j}\Big[w_j\delta_j(f) + \frac{w_j}{w_i}C_{ij}\Delta_w(f)\Big] \\ &\le \Big(|I| - 1 + \sum_{i\neq j}\frac{w_j}{w_i}C_{ij}\Big)\Delta_w(f) \le (|I| - c_w)\,\Delta_w(f). \end{aligned}$$

Here $|I| - c_w \ge |I| - 1 \ge 0$. Iterating, $\Delta_w\big((\sum_i Q_i)^k f\big) \le (|I| - c_w)^k \Delta_w(f)$. The series for $P_s f$ converges in the supremum norm, and $\delta_j(g) \le \delta_j(g_n) + 2\|g - g_n\|_\infty$; hence

$$\Delta_w(P_s f) \le \mathrm{e}^{-|I|s}\sum_k \frac{s^k(|I| - c_w)^k}{k!}\,\Delta_w(f) = \mathrm{e}^{-c_w s}\Delta_w(f).$$

For the last statement, $\sup P_s f - \inf P_s f \le \sum_j \delta_j(P_s f) \le \Delta_w(P_s f)\sum_j w_j^{-1}$, and $\mu(P_s f) = \mu(f)$ lies between the infimum and the supremum. □

The factor $|I|$ cancels exactly in the last display. No quantity depending on the size of the system enters the rate c_w.

Proposition 5.4 (Covariance bound) *Assume $c_w > 0$. For bounded measurable real functions f, g on X,*

$$\big|\mu(fg) - \mu(f)\mu(g)\big| \le \frac{\Delta_w(g)}{2c_w}\sum_{i\in I}\frac{\delta_i(f)}{w_i}.$$

Proof For bounded u, v let $\mathrm{Cov}_i(u,v) = Q_i(uv) - Q_i u\, Q_i v$. By (Q2) and (Q3), $\mu(u\,(v - Q_i v)) = \mu(Q_i(uv)) - \mu(Q_i(u\,Q_i v)) = \mu(\mathrm{Cov}_i(u,v))$. Since $s \mapsto P_s g$ is differentiable in the supremum norm with derivative $LP_s g$,

$$\mu(fg) - \mu(fP_S g) = -\int_0^S \mu(f\,LP_s g)\,\mathrm{d}s = \int_0^S \sum_i \mu\big(\mathrm{Cov}_i(f, P_s g)\big)\,\mathrm{d}s.$$

At a fixed σ, $\mathrm{Cov}_i(u,v)(\sigma)$ is the covariance under μ_i^σ of the functions $\eta \mapsto u(\eta\sigma_{i^c})$ and $\eta \mapsto v(\eta\sigma_{i^c})$. Hence

$$\begin{aligned}|\mathrm{Cov}_i(u,v)(\sigma)| &= \tfrac{1}{2}\Big|\iint \big(u(\eta\sigma_{i^c}) - u(\eta'\sigma_{i^c})\big)\big(v(\eta\sigma_{i^c}) - v(\eta'\sigma_{i^c})\big)\mu_i^\sigma(\mathrm{d}\eta)\mu_i^\sigma(\mathrm{d}\eta')\Big| \\ &\le \tfrac{1}{2}\delta_i(u)\,\delta_i(v).\end{aligned}$$

With $u = f$, $v = P_s g$ and $\delta_i(P_s g) \le \mathrm{e}^{-c_w s}\Delta_w(g)/w_i$ (Proposition 5.3), integration over $s \in [0,\infty)$ bounds the right-hand side by the claimed quantity. Since $\mu(fP_S g) \to \mu(f)\mu(g)$ as $S \to \infty$ (Proposition 5.3), the claim follows. □

5.3 A tilting bound

Lemma 5.5 *Let ν be a probability measure, let h be a bounded measurable real function, and let $\nu_h = \mathrm{e}^h\nu/\nu(\mathrm{e}^h)$. Then*

$$\mathrm{TV}(\nu_h, \nu) \le \tanh\big(\tfrac{1}{4}\operatorname{osc} h\big).$$

Consequently, if $\nu_1 = \mathrm{e}^{h_1}\nu/\nu(\mathrm{e}^{h_1})$ and $\nu_2 = \mathrm{e}^{h_2}\nu/\nu(\mathrm{e}^{h_2})$, then $\mathrm{TV}(\nu_1,\nu_2) \le \tanh\big(\tfrac{1}{4}\operatorname{osc}(h_2 - h_1)\big)$.

Proof Let A be measurable with $0 < p = \nu(A) < 1$ (otherwise there is nothing to prove). Let a and b be the ν-averages of e^h over A and over its complement, and let $R = a/b \in [\mathrm{e}^{-\operatorname{osc} h}, \mathrm{e}^{\operatorname{osc} h}]$. Then

$$\nu_h(A) - \nu(A) = \frac{pR}{pR + 1 - p} - p = \frac{p(1-p)(R-1)}{pR + 1 - p}.$$

Replacing A by its complement changes only the sign, so we may assume $R \ge 1$, and we put $s = \sqrt{R} \ge 1$. The identity

$$ps^2 + 1 - p - p(1-p)(s+1)^2 = \big(p(s+1) - 1\big)^2 \ge 0$$

gives $p(1-p)(s^2-1)/(ps^2+1-p) \le (s-1)/(s+1) = \tanh(\tfrac{1}{4}\log R) \le \tanh(\tfrac{1}{4}\operatorname{osc} h)$. For the second statement apply the first with ν_1 in place of ν and $h = h_2 - h_1$: then $(\nu_1)_h = \nu_2$. □

6 The volume- and rank-uniform transfer gap

Throughout this section $N \ge 2$, $\beta > 0$, Γ is a spatial graph with $\mathsf{E} \ne \varnothing$, $T = T_\beta$, and λ_0, $\Omega = \Omega_\Gamma$ are as in Proposition 4.2; $S = T/\lambda_0$.

6.1 An auxiliary periodic measure and the trace formula

Definition 6.1 For $L_{\mathrm{t}} \ge 3$ let $I_{L_{\mathrm{t}}} = \mathsf{E} \times \mathbb{Z}_{L_{\mathrm{t}}}$, and let $\tilde\mu_{L_{\mathrm{t}}}$ be the probability measure on $X_{L_{\mathrm{t}}} = (G^{\mathsf{E}})^{\mathbb{Z}_{L_{\mathrm{t}}}}$, with points $(U_{e,\tau})$ and $U_\tau = (U_{e,\tau})_{e\in\mathsf{E}}$, given by

$$\tilde\mu_{L_{\mathrm{t}}}(\mathrm{d}U) = \frac{1}{\tilde Z_{L_{\mathrm{t}}}} \prod_{\tau\in\mathbb{Z}_{L_{\mathrm{t}}}} \mathrm{e}^{-\beta\mathcal{V}_\Gamma(U_\tau)} \prod_{\tau\in\mathbb{Z}_{L_{\mathrm{t}}}}\prod_{e\in\mathsf{E}} w_\beta\big(U_{e,\tau}U_{e,\tau+1}^{-1}\big) \prod_{(e,\tau)\in I_{L_{\mathrm{t}}}} \mathrm{d}U_{e,\tau}.$$

The *time* of the site (e,τ) is τ, and $d_{L_{\mathrm{t}}}$ denotes the graph distance on the cycle $\mathbb{Z}_{L_{\mathrm{t}}}$.

As explained in Remark 2.5, $\tilde\mu_{L_t}$ is not the Wilson measure; it is a spin system in the sense of Section 5, with $X_i = G$, m_i the Haar measure, and $H = -\log(\text{density})$. The density is continuous and strictly positive on a compact space, so H is bounded.

Lemma 6.2 (Trace formula) *For bounded measurable $f_0, \dots, f_{L_t-1}$ on G^{E},*

$$\int \prod_{\tau \in \mathbb{Z}_{L_t}} f_\tau(U_\tau)\, \mathrm{d}\tilde\mu_{L_t} = \frac{\operatorname{tr}\bigl(f_0 T\, f_1 T \cdots f_{L_t-1} T\bigr)}{\operatorname{tr}(T^{L_t})}.$$

In particular $\tilde\mu_{L_t}$ is invariant under the shift $\tau \mapsto \tau+1$. For bounded measurable real functions f, g on G^{E} and fixed $n \geq 0$,

$$\lim_{L_t\to\infty} \int f(U_0)\, g(U_n)\, \mathrm{d}\tilde\mu_{L_t} = \langle f\,\Omega, S^n\, g\,\Omega\rangle, \qquad \lim_{L_t\to\infty} \int f(U_0)\, \mathrm{d}\tilde\mu_{L_t} = \langle \Omega, f\,\Omega\rangle.$$

Proof The density of $\tilde\mu_{L_t}$ is $\tilde Z_{L_t}^{-1} \prod_\tau M_\beta(U_\tau)^2 K_\beta(U_\tau, U_{\tau+1})$, where $K_\beta(U,V) = \prod_e w_\beta(U_e V_e^{-1})$ is the kernel of K_β. By Lemma B.1,

$$\int \prod_\tau f_\tau(U_\tau) \prod_\tau M_\beta(U_\tau)^2 K_\beta(U_\tau, U_{\tau+1})\, \mathrm{d}U = \operatorname{tr} \prod_{\tau=0}^{L_t-1} \bigl(M_\beta^2 f_\tau K_\beta\bigr).$$

Since multiplication operators commute, $M_\beta^2 f_\tau K_\beta = M_\beta (f_\tau T) M_\beta^{-1}$. The product telescopes to $M_\beta \bigl(\prod_\tau f_\tau T\bigr) M_\beta^{-1}$, whose trace is $\operatorname{tr}(\prod_\tau f_\tau T)$. Normalizing gives the formula, and shift invariance follows from cyclicity of the trace.

By Proposition 4.2, $T \geq 0$ is trace class with simple top eigenvalue λ_0 and eigenvector Ω. Let $\lambda_0 > \lambda_1' \geq \lambda_2' \geq \dots$ be its other eigenvalues, with multiplicity. Then $R_m := T^m/\lambda_0^m - |\Omega\rangle\langle\Omega| \geq 0$ has trace norm

$$\|R_m\|_1 = \sum_{k\geq 1} (\lambda_k'/\lambda_0)^m \leq (\lambda_1'/\lambda_0)^{m-1} \lambda_0^{-1} \operatorname{tr} T \longrightarrow 0 \qquad (m \to \infty).$$

Hence $|\operatorname{tr}(AT^m)/\lambda_0^m - \langle\Omega, A\Omega\rangle| \leq \|A\|\,\|R_m\|_1 \to 0$ for every bounded operator A. By the formula just proved, $\int f(U_0) g(U_n)\, \mathrm{d}\tilde\mu_{L_t} = \operatorname{tr}(fT^n g T^{L_t-n})/\operatorname{tr}(T^{L_t})$. Apply the limit with $A = fT^n g$, $m = L_t - n$ in the numerator and with $A = 1$, $m = L_t$ in the denominator; since $\langle\Omega, fT^n g\,\Omega\rangle = \lambda_0^n \langle f\Omega, S^n g\Omega\rangle$ for real f, the first limit follows. The second is the case $n = 0$, $g = 1$. □

6.2 Influences

Lemma 6.3 (Single-link influences) *Consider the spin system $\tilde\mu_{L_t}$, $L_t \geq 3$, and sites $i = (e,\tau)$ and $j \neq i$.*

(i) *If $j = (e',\tau)$ with $e' \neq e$, and e, e' lie in exactly $m \geq 1$ common plaquettes, then $C_{ij} \leq \tanh(m\beta) \leq m \tanh\beta$.*

(ii) *If $j = (e, \tau \pm 1)$, then $C_{ij} \leq \tanh\beta$.*

(iii) *In all other cases $C_{ij} = 0$.*

Proof Write $s_N(g) = N^{-1}\operatorname{Re}\operatorname{tr} g$, $u = U_{e,\tau}$ and $V_\pm = U_{e,\tau\pm1}$. Since $L_\mathrm{t} \geq 3$, the two temporal neighbors are distinct. Up to normalization the conditional density with respect to Haar measure is

$$\exp\Big(\beta \sum_{p \ni e} W_p(U_\tau)\Big) w_\beta(uV_+^{-1}) w_\beta(V_- u^{-1}).$$

Each plaquette uses u exactly once. Cyclicity and $s_N(g^{-1}) = s_N(g)$ therefore give $W_p(U_\tau) = s_N(uB_p)$ with B_p a product of the other three variables or their inverses, independent of u. If u occurs inversely, first invert the whole plaquette product. Variables outside the stated neighbors do not occur in this conditional density, proving (iii).

Changing a same-time neighbor shared by m plaquettes changes the logarithmic density by

$$h(u) = \beta \sum_{p \ni e,e'} \big(s_N(uB_p') - s_N(uB_p)\big).$$

Since $|s_N| \leq 1$, each difference belongs to $[-2,2]$, so $h \in [-2m\beta, 2m\beta]$ and $\operatorname{osc} h \leq 4m\beta$. Lemma 5.5 gives $C_{ij} \leq \tanh(m\beta) \leq m \tanh\beta$, where the last inequality follows from concavity of tanh on $[0,\infty)$ and its value zero at the origin.

Changing V_+ changes only $\beta s_N(uV_+^{-1})$ and gives $\operatorname{osc} h \leq 4\beta$, hence $C_{ij} \leq \tanh\beta$. The case V_- is identical. At $L_\mathrm{t} = 3$ there is also a temporal bond between V_+ and V_-, but it is independent of u and cancels from its conditional law. All estimates are independent of the group parameter N. □

Lemma 6.4 (Weighted Dobrushin constant) *Let $\gamma \geq 0$ and $n \in \mathbb{N}_0$, let $L_\mathrm{t} \geq \max(3, 2n)$, and let $w_{(e,\tau)} = \mathrm{e}^{\gamma d_{L_\mathrm{t}}(\tau,n)}$. Then*

$$c_w \;\geq\; \Theta_\gamma := 1 - \tanh\beta\,\big(12 + 2\mathrm{e}^{\gamma}\big).$$

Proof Fix $j = (e',\tau')$. By Lemma 6.3, the sites i with $C_{ij} \neq 0$ are of two kinds. The first are the same-time sites (e,τ'), $e \neq e'$, where e shares $m_e \geq 1$ plaquettes with e'; for these $w_j/w_i = 1$ and $C_{ij} \leq m_e \tanh\beta$. Every plaquette containing e' has three other edges, and e' lies in at most four plaquettes, so $\sum_e m_e \leq 12$. The second are the two sites $(e',\tau' \pm 1)$; for these $w_j/w_i = \mathrm{e}^{\gamma(d_{L_\mathrm{t}}(\tau',n) - d_{L_\mathrm{t}}(\tau'\pm1,n))} \leq \mathrm{e}^{\gamma}$ by the triangle inequality on $\mathbb{Z}_{L_\mathrm{t}}$, and $C_{ij} \leq \tanh\beta$. Summing,

$$\sum_{i \neq j} \frac{w_j}{w_i} C_{ij} \leq 12 \tanh\beta + 2\mathrm{e}^{\gamma} \tanh\beta = 1 - \Theta_\gamma.$$

□

The constant Θ_γ depends only on γ, on β, and on the number 4 in Definition 2.1; it does not depend on N, Γ, L_t or n. The weight ratio bound e^{γ} is attained for both temporal neighbours at once: when L_t is even, at the time slice antipodal to n both temporal neighbours are closer to n. This is why the factor is $2\mathrm{e}^{\gamma}$ and not $\mathrm{e}^{\gamma} + \mathrm{e}^{-\gamma}$.

6.3 Decay of time correlations in the ground state

Proposition 6.5 *Let $\gamma \geq 0$ with $\Theta_\gamma > 0$. For every bounded measurable real function b on G^{E} and every $n \geq 0$,*

$$\big|\langle b\,\Omega, S^n\, b\,\Omega\rangle - \langle \Omega, b\,\Omega\rangle^2\big| \leq \frac{2\,|\mathsf{E}|\,\|b\|_\infty^2}{\Theta_\gamma}\,\mathrm{e}^{-\gamma n}.$$

Proof Let $L_{\mathrm{t}} \geq \max(3, 2n)$ and apply Proposition 5.4 to $\tilde{\mu}_{L_{\mathrm{t}}}$ with $f(U) = b(U_0)$, $g(U) = b(U_n)$ and the weight of Lemma 6.4. The function g depends only on the sites (e, n), where $w = 1$; hence $\Delta_w(g) \leq 2\|b\|_\infty$. The function f depends only on the sites $(e, 0)$, where $w = \mathrm{e}^{\gamma d_{L_{\mathrm{t}}}(0,n)} = \mathrm{e}^{\gamma n}$ because $n \leq L_{\mathrm{t}}/2$; hence $\sum_i \delta_i(f)/w_i \leq 2\|b\|_\infty |\mathsf{E}| \mathrm{e}^{-\gamma n}$. Since $c_w \geq \Theta_\gamma > 0$,

$$\Big| \int b(U_0) b(U_n) \, \mathrm{d}\tilde{\mu}_{L_{\mathrm{t}}} - \int b(U_0) \, \mathrm{d}\tilde{\mu}_{L_{\mathrm{t}}} \int b(U_n) \, \mathrm{d}\tilde{\mu}_{L_{\mathrm{t}}} \Big| \leq \frac{2|\mathsf{E}| \, \|b\|_\infty^2}{\Theta_\gamma} \mathrm{e}^{-\gamma n}$$

for every $L_{\mathrm{t}} \geq \max(3, 2n)$. By shift invariance $\int b(U_n) \, \mathrm{d}\tilde{\mu}_{L_{\mathrm{t}}} = \int b(U_0) \, \mathrm{d}\tilde{\mu}_{L_{\mathrm{t}}}$. Letting $L_{\mathrm{t}} \to \infty$ at fixed n and using Lemma 6.2 gives the claim. □

6.4 The eigenfunction-ratio test

Proposition 6.6 *Let $\gamma \geq 0$ with $\Theta_\gamma > 0$. Then every eigenvalue $\lambda \neq \lambda_0$ of T on $\mathcal{H}_{\mathrm{kin}}$ satisfies $\lambda \leq \mathrm{e}^{-\gamma} \lambda_0$.*

Proof By Proposition 4.2(iv), λ has a real eigenfunction ψ with $\|\psi\| = 1$ and $\psi \perp \Omega$. By Proposition 4.2(ii),(iii), ψ and Ω are continuous and $\min \Omega > 0$. Therefore $b := \psi/\Omega$ is a bounded continuous real function, with $b\,\Omega = \psi$ and $\langle \Omega, b\,\Omega \rangle = \langle \Omega, \psi \rangle = 0$. Moreover $\langle b\Omega, S^n b\Omega \rangle = \langle \psi, S^n \psi \rangle = (\lambda/\lambda_0)^n$, and $\lambda > 0$ by Proposition 4.2(iv). Proposition 6.5 gives

$$0 < \Big(\frac{\lambda}{\lambda_0}\Big)^n \leq C_\Gamma \, \mathrm{e}^{-\gamma n} \quad \text{for all } n \geq 1, \qquad C_\Gamma = \frac{2|\mathsf{E}| \, \|\psi/\Omega\|_\infty^2}{\Theta_\gamma}.$$

The constant C_Γ depends on N, Γ, and ψ but not on n. Hence $\lambda/\lambda_0 \leq C_\Gamma^{1/n} \mathrm{e}^{-\gamma}$ for every $n \geq 1$, and letting $n \to \infty$ gives $\lambda/\lambda_0 \leq \mathrm{e}^{-\gamma}$. □

6.5 Proof of Theorem 1.1

Proof of Theorem 1.1 If $\mathsf{E} = \varnothing$, the assertions follow from $T_\beta = 1$ on $\mathbb{C}$. Assume $\mathsf{E} \neq \varnothing$. Part (i) is Proposition 4.2(iii). Let $0 < \beta < \beta_{\mathrm{W}}$, i.e. $0 < \tanh\beta < \frac{1}{14}$. Then $0 < 2\tanh\beta < 1 - 12\tanh\beta$, which is equivalent to $0 < q(\beta) < 1$. Put

$$x_* = \frac{1 - 12\tanh\beta}{2\tanh\beta} = \frac{1}{q(\beta)} > 1.$$

For every $\gamma \geq 0$ with $\mathrm{e}^\gamma < x_*$ we have $\Theta_\gamma = 1 - \tanh\beta(12 + 2\mathrm{e}^\gamma) > 1 - \tanh\beta(12 + 2x_*) = 0$. Proposition 6.6 then gives $\lambda \leq \mathrm{e}^{-\gamma}\lambda_0$ for every eigenvalue $\lambda \neq \lambda_0$ of T_β on $\mathcal{H}_{\mathrm{kin}}$. Letting $\mathrm{e}^\gamma \uparrow x_*$ gives $\lambda \leq x_*^{-1}\lambda_0 = q(\beta)\lambda_0$. By Proposition 4.2(iv), $\sigma(T_\beta) \setminus \{\lambda_0\}$ consists of these eigenvalues, which are positive, and possibly the point 0. This proves (ii) on $\mathcal{H}_{\mathrm{kin}}$. Since P_G commutes with T_β and $\Omega_\Gamma \in \mathcal{H}_{\mathrm{phys}}$, the space $\mathcal{H}_{\mathrm{phys}} \ominus \mathbb{C}\Omega_\Gamma$ reduces T_β and is contained in $\mathcal{H}_{\mathrm{kin}} \ominus \mathbb{C}\Omega_\Gamma$; hence $\lambda_1(\Gamma, \beta) \leq \lambda_1^{\mathrm{kin}}(\Gamma, \beta) \leq q(\beta)\lambda_0$. Part (iii) follows because $q(\beta)$ depends on neither N nor Γ. □

Remark 6.7 (Optimization in γ) The supremum of the admissible rates γ is $\log x_* = \log(1/q(\beta))$. It is not attained (at $\mathrm{e}^\gamma = x_*$ one has $\Theta_\gamma = 0$), so $q(\beta)$ is obtained as a limit of the bounds $\mathrm{e}^{-\gamma}$. The threshold β_{W} is the condition $x_* > 1$, i.e. that some $\gamma > 0$ is admissible; it is the same as $\Theta_0 = 1 - 14\tanh\beta > 0$.

7 Discussion and limitations

What has been proved.

For the Wilson $SU(N)$ lattice gauge theory at fixed lattice spacing and $\tanh\beta < 1/14$, every finite-volume transfer operator, on every spatial graph with at most four plaquettes per edge, has its spectrum on the orthogonal complement of the ground state contained in $[0, q(\beta)\lambda_0]$, with an explicit $q(\beta) < 1$ depending on β alone, simultaneously for every integer $N \geq 2$. This holds on the full kinematical space, hence on the gauge-invariant space, without a representation cutoff and without gauge fixing.

Fixed lattice spacing and strong coupling.

All statements are at fixed lattice spacing. The region $\beta < \operatorname{artanh}(1/14)$ is a region of very strong coupling, and the estimate is crude: we make no claim that β_{W} or $q(\beta)$ is close to optimal. The single-link influences are themselves of order β, and we do not expect the single-link condition to hold much beyond this region; we do not prove a failure threshold. Reaching larger β by the same route would require a block condition of Dobrushin–Shlosman type, whose validity for gauge theories at intermediate or weak coupling is not addressed here.

No continuum statement.

Nothing in this paper concerns the continuum limit $a \to 0$. The gap bound is per lattice time step; in physical units it reads $m \geq a^{-1}\log(1/q(\beta))$ at isotropic spacing a, which gives no information along a sequence $a \to 0$, $\beta \to \infty$. In particular the theorem gives no information about a mass gap of a continuum Yang–Mills theory.

Rank uniformity and realification.

The influence estimate uses only $|N^{-1}\operatorname{Re}\operatorname{tr} g| \leq 1$ and the graph incidence bound. The representation-theoretic input establishes strict positivity and trace class separately for each rank, while all rank-dependent ground-state prefactors disappear after n-th roots. The theorem therefore holds for every $SU(N)$ at fixed normalized-action β. It does not assert a gap uniform in N at fixed $b = \beta/N^2$. The CRE correspondence with the image subgroup of $SO(2N)$ preserves this Wilson theory with its transported Haar measure and exact trace normalization (Section C); it is an equivalence of models, not a substitute for the analytic estimates. Other compact gauge groups are not treated here.

Infinite volume.

We make no infinite-volume statement. The finite-volume bound concerns each operator T_Γ separately. Passing to infinite volume would require constructing limit states and their reconstructed Hilbert spaces, and identifying which finite-volume vectors survive the limit; we do not carry this out here. Exponential decay of correlations for the infinite-volume Wilson measure at strong coupling has been proved by Shen, Zhu and Zhu [5]. The result of the present paper is of a different kind: it controls the spectrum of every finite-volume transfer operator, including states that need not have a counterpart in a given infinite-volume representation (Section 1.2).

Statements and Declarations

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Data availability. This is a theoretical study and uses no empirical datasets. The numerical values in Appendix B are obtained from the formulas stated there; computational checks support, but do not replace, the mathematical proofs.

Use of AI-assisted tools. OpenAI Codex and Claude were used to assist with mathematical verification, consistency checks, literature review, and manuscript preparation, including drafting and typesetting. All mathematical results and theoretical formulations are attributed to the author, who reviewed the final arguments and assumes full responsibility for the manuscript.

Lean was used to check selected auxiliary results concerning rank-one projections, quadratic-form implications, and scalar logarithmic gap inequalities. These checks do not constitute a formal verification of Theorem 1.1.

Appendix A Wilson characters for $SU(N)$

Let $\widehat{G}$ denote the irreducible unitary representations of $G = SU(N)$, and write $d_\pi = \dim \pi$. The Peter–Weyl theorem decomposes $L^2(G)$ into their matrix-coefficient spaces of dimensions d_π^2. Schur orthogonality gives convolution by χ_π as d_π^{-1} on the corresponding space and zero on the others. In general χ_π need not be real. The expansion below uses both the defining representation and its conjugate.

Lemma A.1 (Wilson character expansion) *Let $N \geq 2$, $\beta > 0$, $V = F \oplus \overline{F}$, and $a = \beta/(2N)$. If $m_{k,\pi}$ is the multiplicity of π in $V^{\otimes k}$, then*

$$\mathrm{e}^{(\beta/N)\,\mathrm{Re}\,\mathrm{tr}\,g} = \sum_{\pi\in\widehat{G}} c_\pi \chi_\pi(g), \qquad c_\pi = \sum_{k\geq 0} \frac{a^k}{k!} m_{k,\pi} > 0, \qquad \sum_\pi d_\pi c_\pi = \mathrm{e}^\beta.$$

The expansion converges absolutely and uniformly, and $z_\beta = c_{\mathbf{1}} \geq 1$.

Proof This is the tensor-character argument used by Lüscher [2, proof of Proposition 1]; we include the occurrence and convergence details. Since $\chi_V = 2\,\mathrm{Re}\,\mathrm{tr}$, the exponential power series is

$$\mathrm{e}^{a\chi_V(g)} = \sum_{k\geq 0} \frac{a^k}{k!} \chi_{V^{\otimes k}}(g).$$

Complete reducibility gives nonnegative integer multiplicities and $\sum_\pi m_{k,\pi} d_\pi = (2N)^k$. Thus Tonelli's theorem yields

$$\sum_{k,\pi} \frac{a^k}{k!} m_{k,\pi} d_\pi = \sum_{k\geq 0} \frac{\beta^k}{k!} = \mathrm{e}^\beta.$$

Because $|\chi_\pi(g)| \leq d_\pi$, this is a uniform absolute majorant for the double character series. Reordering gives the asserted expansion and convergence. Integration gives $z_\beta = c_{\mathbf{1}} \geq 1$, the last inequality from the $k = 0$ term.

For strict positivity, consider the complex span $\mathcal{A}$ of matrix coefficients of all $V^{\otimes k}$, including constants at $k = 0$. Products of coefficients are coefficients of tensor products, so $\mathcal{A}$ is an algebra. It is closed under conjugation because V is equivalent to its conjugate. It separates group elements since it contains the matrix entries of the faithful defining representation. Stone–Weierstrass makes $\mathcal{A}$ uniformly dense in $C(G)$, hence dense in $L^2(G)$. If an irreducible π occurred in no tensor power, its nonzero matrix coefficients would be orthogonal to $\mathcal{A}$ by Schur orthogonality, a contradiction. Therefore some $m_{k,\pi} > 0$, and $c_\pi > 0$ because $a > 0$. □

The normalized density is continuous, strictly pointwise positive, central and inversion invariant, since $\operatorname{Re}\operatorname{tr}(g^{-1}) = \operatorname{Re}\operatorname{tr} g$. Its positive definiteness is a separate consequence of the positive character coefficients. When $N = 2$, F and $\overline{F}$ are equivalent, and their two copies must both be retained in V. As a check, for spin $j \in \frac{1}{2}\mathbb{N}_0$, Weyl integration gives $c_j = I_{2j}(\beta) - I_{2j+2}(\beta) = 2(2j+1)I_{2j+1}(\beta)/\beta$, by the Bessel recurrence [16, eq. 10.29.1]. Hence $\kappa_j = I_{2j+1}(\beta)/I_1(\beta)$, recovering the $SU(2)$ formula.

Appendix B A trace lemma and numerical values

Lemma B.1 *Let (Y, m) be a probability space. For $\tau = 0, \dots, L_{\mathrm{t}} - 1$, with $L_{\mathrm{t}} \geq 2$, let A_τ be the operator on $L^2(Y, m)$ with bounded measurable kernel $a_\tau(x, y)$. Then $A_0 A_1 \cdots A_{L_{\mathrm{t}}-1}$ is trace class, and*

$$\operatorname{tr}(A_0 \cdots A_{L_{\mathrm{t}}-1}) = \int_{Y^{L_{\mathrm{t}}}} \prod_{\tau=0}^{L_{\mathrm{t}}-1} a_\tau(x_\tau, x_{\tau+1}) \prod_\tau m(\mathrm{d}x_\tau), \qquad x_{L_{\mathrm{t}}} := x_0.$$

Proof Each A_τ is Hilbert–Schmidt, because its kernel is bounded and m is finite. The product $B = A_1 \cdots A_{L_{\mathrm{t}}-1}$ is Hilbert–Schmidt, and by Fubini's theorem its kernel is

$$b(x_1, x_0) = \int \prod_{\tau=1}^{L_{\mathrm{t}}-1} a_\tau(x_\tau, x_{\tau+1})\, m(\mathrm{d}x_2) \cdots m(\mathrm{d}x_{L_{\mathrm{t}}-1}) \qquad (x_{L_{\mathrm{t}}} = x_0).$$

The product of two Hilbert–Schmidt operators is trace class, and

$$\operatorname{tr}(A_0 B) = \iint a_0(x_0, x_1)\, b(x_1, x_0)\, m(\mathrm{d}x_0)\, m(\mathrm{d}x_1).$$

Substituting gives the formula. □

In this paper the lemma is applied with $Y = G^{\mathsf{E}}$ and $A_\tau = M_\beta^2 f_\tau P_G K_\beta$ (Proposition 2.4) or $A_\tau = M_\beta^2 f_\tau K_\beta$ (Lemma 6.2); the kernels are bounded because f_τ is bounded and w_β is continuous.

B.1 Numerical values

The values in Table 2 were recomputed with 60-digit arithmetic. They are illustrations; the proofs use only the stated inequalities. The threshold is $\beta_{\mathrm{W}} = \operatorname{artanh}\frac{1}{14} = \frac{1}{2}\log\frac{15}{13} = 0.0715504\ldots$. The function $q(\beta) = 2\tanh\beta/(1 - 12\tanh\beta)$ is increasing on $(0, \beta_{\mathrm{W}})$, with $q(\beta) \to 1$ as $\beta \uparrow \beta_{\mathrm{W}}$ and $q(\beta) = 2\beta + 24\beta^2 + O(\beta^3)$ as $\beta \downarrow 0$.

β	0.005	0.01	0.02	0.03	0.04	0.05	0.06	0.07
$q(\beta)$	0.01064	0.02273	0.05262	0.09371	0.1537	0.2495	0.4267	0.8662
$\log(1/q(\beta))$	4.543	3.784	2.945	2.368	1.873	1.388	0.8516	0.1437

Table 2 Spectral-ratio bound and logarithmic gap lower bound, valid for every $N \geq 2$ and every admissible spatial graph.

Appendix C Transport by the CRE correspondence

The author's AXIOM–CRE correspondence [17] gives the following exact coordinate realization. We state its consequence for Wilson theory to distinguish algebraic transport from a spectral estimate.

Proposition C.1 (Realified Wilson theory) *For $Z = A + iB$ put*

$$\Phi_N(Z) = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

The group $G_R = \Phi_N(SU(N))$ is the subgroup of $SO(2N)$ commuting with J and having complex determinant one. With normalized Haar on G_R and observable $(2N)^{-1}\operatorname{tr}_{\mathbb{R}} V_p$, its Wilson transfer is unitarily equivalent to $T_{\beta,N}$, including the gauge projection.

Proof Block multiplication gives $\Phi(ZW) = \Phi(Z)\Phi(W)$ and $\Phi(Z^*) = \Phi(Z)^T$. The map is injective, preserves inverses, and satisfies $\operatorname{tr}_{\mathbb{R}} \Phi(Z) = 2\operatorname{Re}\operatorname{tr}_{\mathbb{C}} Z$. A real matrix commuting with J has the displayed block form; orthogonality is then equivalent to unitarity. The determinant-one condition over $\mathbb{C}$ selects $SU(N)$ from $U(N)$. The continuous isomorphism onto the compact image transports normalized Haar by its defining invariance and uniqueness. Applying it linkwise preserves path products, gauge actions and

$$\frac{1}{2N}\operatorname{tr}_{\mathbb{R}} \Phi(U_p) = \frac{1}{N}\operatorname{Re}\operatorname{tr}_{\mathbb{C}} U_p.$$

Consequently the unitary map $f \mapsto f \circ \Phi^{-1}$ intertwines multiplication, convolution, gauge averaging and their transfer product. □

The measure in this proposition is Haar on G_R, not on all of $SO(2N)$. The generator, connection and curvature identities and the signed-XOR compiler in [17] preserve the same products; none gives a contraction estimate by itself. The analytic gap still requires Sections 4 to 6.

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