# FINER: Filtered Infinitesimal Nets for Energetic Recovery

A Standard-Analytic Theory of Conditional Variational Limits

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**Abstract**

Conditional expectation contracts the state norm but can amplify a closed quadratic energy. For a densely defined closed nonnegative form vanishing on constants, exact contraction under every binary observation forces the energy to be a multiple of variance. We prove that this rigidity has no universal positive-tolerance stability theorem. For every positive tolerance, the fixed Lebesgue probability space admits a closed, densely defined energy whose binary conditioning costs lie within that tolerance of one, although its operator is unbounded, its form domain is proper, and its positive spectrum has infimum zero. A countable atomic construction has the same properties and is also a Dirichlet form. We obtain sharp stability constants for arbitrary three-atom probabilities, a dimension obstruction with equiprobable atoms, and corrected finite-dimensional bounds from observation commutators. Independent spectral regularizations preserve the binary costs and converge in norm resolvent sense, while the limiting obstruction persists. Within FINER, meaning Filtered Infinitesimal Nets for Energetic Recovery, we retain the specified response operators rather than only their scalar energies. Generalized strong resolvent convergence yields an exact response criterion in terms of active polar orientations and adjoint recovery tests. Independent long-time damping is characterized by convergence of the stationary projections; prescribed coupled rates are characterized by moving spectral windows. Complete finite-combination observation testing characterizes uniform inverse response; uniform individual event tests still fail. All constructions use standard analysis and remain independent of unresolved field theories.

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## 1 Introduction

A nonnegative closed form measures energy in a Hilbert space. An orthogonal observation projection preserves the state norm bound, but need not preserve the form domain or contract the energy. For a densely defined form with zero energy on constants, requiring contraction under every binary conditional expectation is particularly restrictive: the associated operator must be scalar on the mean-zero subspace. This article asks what survives when the contractions are only approximate, and what additional data identify responses and stationary modes under variational refinement.

The first question has a negative answer in a strong sense. For every $\varepsilon > 0$, Theorem 5.2 constructs a single form on the fixed space $L^2(0,1)$ for which

$$q(P_B u) \le (1+\varepsilon) q(u) \quad \text{for all binary events and all } u \in \operatorname{Dom} q,$$

yet its domain is proper, its operator unbounded, and its mean-zero spectral lower edge zero. Every centered indicator is in both the forward and inverse half-domains. Theorem 4.1 gives a separate countable atomic example with the Markov contraction property. Thus neither nonatomicity, all-binary testing, nor the Dirichlet-form condition by itself repairs approximate scalar rigidity. The kernel alternative nevertheless survives: a nonzero form under finite binary costs has no zero-energy directions beyond constants.

Theorem 14.1 identifies exactly what upgrades observation recovery to uniform inverse response: control of every finite linear combination, or equivalently the full response Gram matrices. Corollary 14.2 shows that even uniform forward and inverse bounds for each individual event do not suffice. This closed-operator criterion is classical; its failure within the same all-event construction is the substantive distinction.

The quantitative correction must retain observation geometry. Theorem 3.1 gives the exact three-atom coefficient

$$C(p_1, p_2, p_3) = \frac{p_{\min}}{4(1 - p_{\max})(1 - p_{\mathrm{mid}})}.$$

It recovers the earlier 3/16 at equal probabilities and identifies the precise rare-event degeneration. Equal atom weights do not give a dimension-free replacement: the harmonic example in Proposition 3.3 hides a fixed or diverging condition number from all binary tests as dimension grows. Theorem 3.5 supplies a constructive positive result using the spectral gap of a finite observation commutator form. Its data depend only on the observations, not on the unknown energy.

The next issue is whether increasingly strong variational recovery can restore the lost conclusion. Theorem 8.1 shows that bounded coercive approximations can preserve every binary cost, converge in norm resolvent sense, and recover each specified square-root response, while their limit remains unbounded and gapless. The two regularization parameters refine independently. A uniform physical or analytic coercivity anchor therefore cannot be inferred from the filtered convergence structure alone.

For actual responses $D_i$ with $q_i = \|D_i \cdot\|^2$, the scalar forms do not determine orientation in the response Hilbert space. Theorem 9.1 gives five equivalent criteria under generalized strong resolvent convergence: preservation of every energetic recovery, one-shift weak or strong response convergence, convergence of the active polar factors, and recovery of a dense set of active adjoint tests. It allows both finite and limiting form domains to be nondense in an ambient space. The distinction is exact: full response graph convergence is stronger than necessary, while unanchored Gram data are weaker than identified response convergence.

Finally, Theorems 11.1 and 11.2 distinguish arbitrary independent long-time refinement from prescribed coupled rates. The former is equivalent to strong convergence of the stationary projections. The latter is controlled by spectral mass above thresholds proportional to the inverse cumulative time. Actual response damping to zero requires no orientation hypothesis, but does not ensure that stationary states survive. The original conditional monotone damping theorem is retained as a corollary.

### 1.1 Established mechanisms and the scope of the contribution

FINER abbreviates *Filtered Infinitesimal Nets for Energetic Recovery*, an author-developed framework formulated in standard analysis. The author-developed ALH and SPADES viewpoints motivate retaining observation directions, commutation defects, conditional covariance and all cross terms. Those viewpoints do not replace the quantitative estimates proved here. No algebraic claim from AXIOM is needed as an assumption of the present analytical theorems.

Directed convergence is classical [21]. Germs and asymptotic algebras also predate this construction [8, 24]. The scalar closed-form limit is within established monotone and Mosco theory [13, 15, 22, 25]. Exact binary rigidity uses the classical reducing-form mechanism, with the elementary irreducibility of centered indicators; finite rank-one irreducibility has a developed literature [17, 20]. The observation cost is a Kantorovich expression [11, 26]. The harmonic profile and indicator estimate used in Proposition 3.3 occur explicitly in Lev's logarithmic norm-separation construction [16, Section 1.3]. Signed duplication and the rank-one cost identity convert that construction to binary conditioning. We claim no new harmonic-number principle. The infinite-dimensional constructions additionally require simultaneous forward and inverse form-domain estimates for every event.

Conditional expectations already identify approximation spaces in the Wasserstein-diffusion construction of Andres and von Renesse [3], under Markov uniqueness. Our specified-response completion does not introduce conditional variational convergence.

Actual gradients and derivations have also been shown to converge under suitable Mosco and geometric hypotheses [2]; polar convergence in conditional expectation settings is older [1]. Unequal-step semigroup approximation and joint discretizations are established subjects [5, 14]. The present contributions are the displayed closed-form obstructions, their precise finite geometric corrections, the binary-cost preserving recovery construction, and the exact operator characterizations. The supplement compares statements and implications theorem by theorem. It does not infer global priority from unsuccessful searches. The germ algebras, directed sets and scalar form limits are classical; conditional recovery can also be obtained by applying monotone-form approximation to the specified response channels. FINER records these channels and their joint control, and does not define a convergence theory stronger than classical graph-augmented Mosco methods.

## 2 Rigidity and the cost of conditional projection

The following established result from the author's FINER investigation explains why an arbitrary closed derivative energy cannot be required to contract under every conditional expectation. Its mechanism is the classical reducing-form criterion and irreducibility of the observation projections, rather than a new rigidity principle for arbitrary operators. The proof is independent of the recovery construction.

**Theorem 2.1** (Universal binary-conditioning rigidity)**.** *Let $(\Omega, \mathcal{A}, \mu)$ be a probability space and let $q$ be a closed, densely defined, nonnegative quadratic form on the real Hilbert space $H = L^2(\mu)$. Write $V = \operatorname{Dom} q$ and assume $1 \in V$ and $q(1) = 0$. For an event $A$ with $0 < \mu(A) < 1$, let $P_A = \mathbb{E}[\cdot \mid \sigma(A)]$. The following are equivalent:*

1. *For every such $A$, $P_A V \subset V$ and $q(P_A u) \le q(u)$ for every $u \in V$.*
2. *There is a finite $c \ge 0$ such that $V = H$ and $q(u) = c\|u - \mathbb{E}u\|^2$ for every $u \in H$.*

*If $H \ominus \mathbb{R}1 \neq \{0\}$, the constant $c$ is unique. No nonatomicity or countable-generation assumption is required.*

*Proof.* Let $a$ be the symmetric bilinear form associated with $q$. Positivity and $q(1) = 0$ give $a(1, u) = 0$ for $u \in V$ by the Cauchy–Schwarz inequality for nonnegative forms. Put $v_A = 1_A - \mu(A)$, so $\|v_A\|^2 = \mu(A)(1 - \mu(A)) > 0$ and

$$P_A u = \langle u, 1\rangle 1 + \frac{\langle u, v_A\rangle}{\|v_A\|^2} v_A. \tag{1}$$

Density of $V$ implies that $\langle u, v_A\rangle \neq 0$ for some $u \in V$. Domain invariance and (1) therefore imply $v_A \in V$.

If $x \in V \cap \operatorname{ran} P_A$ and $y \in V \cap \ker P_A$, contraction applied to $x + ty$ gives

$$0 \leq 2t\, a(x, y) + t^2 q(y) \qquad (t \in \mathbb{R}).$$

Hence $a(x, y) = 0$. Decomposing $u \in V$ with $P_A$ now gives

$$a(v_A, u) = \lambda_A \langle v_A, u\rangle, \qquad \lambda_A = \frac{q(v_A)}{\|v_A\|^2} \geq 0.$$

Let $L$ be the nonnegative self-adjoint operator associated with $q$. The displayed identity is precisely the defining operator-domain condition, so $v_A \in \operatorname{Dom} L$ and $Lv_A = \lambda_A v_A$. Also $1 \in \operatorname{Dom} L$ and $L1 = 0$.

Self-adjointness yields $(\lambda_A - \lambda_B)\langle v_A, v_B\rangle = 0$. If the covariance is nonzero, then $\lambda_A = \lambda_B$. If it is zero, let $C = A \cap B$. Then $\mu(C) = \mu(A)\mu(B) \in (0, 1)$, and

$$\langle v_A, v_C\rangle = \mu(C)(1 - \mu(A)) > 0, \qquad \langle v_B, v_C\rangle = \mu(C)(1 - \mu(B)) > 0.$$

Thus $\lambda_A = \lambda_C = \lambda_B$ in this case as well. If the mean-zero subspace is nonzero, there is a nontrivial event; fix one and call its eigenvalue $c$. All centered indicators have that eigenvalue. Consequently $L = c(I - \mathbb{E})$ on the dense space of simple functions. For any $u \in H$, choose simple $u_n \to u$. Then $Lu_n = c(I - \mathbb{E})u_n \to c(I - \mathbb{E})u$. Closedness of the operator graph gives $u \in \operatorname{Dom} L$ and the same formula for $Lu$. In particular $L$ is bounded, its form domain is all of $H$, and $q(u) = c\|u - \mathbb{E}u\|^2$. If the mean-zero subspace is zero, density and $1 \in V$ give $V = H$ and $q = 0$ directly. Conversely, conditional expectation preserves the mean and contracts the $L^2$ norm, which proves the required inequality for the form in (2). Uniqueness follows by testing a nonzero mean-zero vector. □

The passage through the closed operator graph in this proof is essential: density of simple functions in $H$ alone would not establish that they form a core for an arbitrary closed form. The theorem concerns all binary observations. An unbounded energy can still be contractive under a specified increasing filtration, as the application below shows.

### 2.1 A sharp certificate for a specified observation

Work on the mean-zero subspace $H_0 = H \ominus \mathbb{R}1$. Let $L \geq 0$ be the associated operator there, and let $v \in \operatorname{Dom} L^{1/2}$ be nonzero with $q(v) > 0$. Define

$$P_v u = \frac{\langle u, v\rangle}{\|v\|^2} v, \qquad K(v) = \inf\{K \geq 0 : q(P_v u) \leq K q(u) \text{ for all } u \in \operatorname{Dom} q\}.$$

The infimum of an empty set is $+\infty$.

**Proposition 2.2** (Binary conditional cost)**.** *With the preceding assumptions, $K(v) < \infty$ if and only if $v \perp \ker L$ and*

$$\int_{(0,\infty)} \lambda^{-1}\, d\|E_L(\lambda)v\|^2 < \infty. \tag{2}$$

*In this case*

$$K(v) = \frac{\|L^{1/2}v\|^2 \|L^{-1/2}v\|^2}{\|v\|^4} \geq 1. \tag{3}$$

*Equality holds if and only if $v$ is an eigenvector of $L$ with a strictly positive eigenvalue. If $q(v) = 0$, then $q(P_v u) = 0$ for all $u$ and the best constant is zero instead. For $v = v_A$, formula (3) is the exact energetic cost of conditional expectation onto the event $A$, after removing constants.*

*Proof.* Direct substitution gives

$$q(P_v u) = \frac{q(v)}{\|v\|^4} |\langle u, v\rangle|^2.$$

The functional $u \mapsto \langle u, v\rangle$ is bounded in the seminorm $\|L^{1/2}u\|$ exactly when it vanishes on $\ker L$ and is represented in the energy completion. Spectral calculus identifies this condition with (2), with squared dual norm $\|L^{-1/2}v\|^2$. More explicitly, under that condition,

$$|\langle u, v\rangle| = |\langle L^{1/2}u, L^{-1/2}v\rangle| \le \|L^{1/2}u\| \|L^{-1/2}v\|.$$

The spectral test vectors $u_{\eta,R} = L^{-1}1_{[\eta,R]}(L)v$ show sharpness as $\eta \downarrow 0$, $R \uparrow \infty$. The same tests give an unbounded quotient if the integral diverges; a nonzero kernel component already makes the functional nonzero on a zero-energy vector. This proves the formula and both finiteness directions.

Since $v \perp \ker L$, $\|v\|^2 = \langle L^{1/2}v, L^{-1/2}v\rangle$. Cauchy–Schwarz proves $K(v) \ge 1$. Its equality condition says that $L^{1/2}v = cL^{-1/2}v$ for some $c > 0$, which by spectral calculus means that the spectral measure of $v$ is supported at $\{c\}$. This is exactly the positive-eigenvalue condition. Its converse and the zero-energy case follow directly. □

For a concrete sharp example on a two-dimensional mean-zero space, let $Le_1 = \lambda e_1$, $Le_2 = \Lambda e_2$, with $0 < \lambda \le \Lambda$, and take $v = \cos\theta\, e_1 + \sin\theta\, e_2$. Then

$$K(v) = 1 + \frac{(\Lambda - \lambda)^2}{\lambda\Lambda} \sin^2\theta \cos^2\theta. \tag{4}$$

At $\theta = \pi/4$ this is $(\Lambda+\lambda)^2/(4\lambda\Lambda)$, the classical Kantorovich extremal factor [11]. The example is realized on a probability space with three equiprobable atoms: choose a normalized centered indicator as $v$, complete it to an orthonormal basis of $H_0$, and choose the eigenbasis of $L$ at angle $\pi/4$ to $v$. Thus the constant is an actual binary-conditioning constant, not merely an arbitrary projection example. In particular it can diverge while the observation remains an $L^2$ contraction.

## 2.2 An exact variational correction

**Proposition 2.3** (Conditional and energy projections)**.** *Let $q$ be a closed nonnegative form on $H$ with domain $V$, and let $P$ be an orthogonal projection in $H$ with $PV \subset V$. For $z > 0$, give $V$ the inner product $a_z(u, v) = a(u, v) + z\langle u, v\rangle$. Then $W = PV = V \cap \operatorname{ran} P$ is closed in this Hilbert space. Let $R_z$ be the $a_z$-orthogonal projection onto $W$. For every $u \in V$, the correction $h = R_z u - Pu$ is uniquely characterized by*

$$a_z(h, v) = a((I - P)u, v), \qquad v \in W, \tag{5}$$

*and satisfies the exact identity*

$$\|R_z u - Pu\|_{a_z} = \|a((I - P)u, \cdot)|_W\|_{(W,a_z)^*}. \tag{6}$$

*Moreover*

$$\|u - Pu\|_{a_z}^2 = \|u - R_z u\|_{a_z}^2 + \|R_z u - Pu\|_{a_z}^2. \tag{7}$$

*Thus conditional and variational projection coincide at $u$ if and only if the displayed cross-energy functional vanishes.*

*Proof.* If $u_n \to u$ in $V$ and $Pu_n \to w$ in $V$, the continuous inclusion into $H$ and boundedness of $P$ in $H$ give $w = Pu$. The closed graph theorem makes $P : V \to V$ bounded. As a bounded idempotent it has closed range $W$. The defining variational identity for $R_z$ gives $a_z(R_z u - Pu, v) = a_z((I - P)u, v)$ for $v \in W$. The $H$ inner product on the right is zero, proving (5). Riesz representation gives (6). Finally $u - Pu = (u - R_z u) + (R_z u - Pu)$ is an $a_z$-orthogonal decomposition, which proves (7). □

No independent cross term is discarded in this result. In particular, $L^2$ orthogonality of $Pu$ and $(I - P)u$ does not imply their $q$ orthogonality. The exact correction is a classical Riesz construction; its role here is to give a verifiable interface for a chosen filtration, rather than to impose the universal contractivity excluded by Theorem 2.1.

# 3 Sharp finite observation geometry

The binary cost in Lemma 4.2 has a finite-dimensional counterpart that is useful independently of the obstruction. All spaces in this section are real and finite-dimensional. On a finite probability space let $H_0 = \{u : \mathbb{E}u = 0\}$, and extend an energy operator on $H_0$ to the form $q(u) = \langle L(u - \mathbb{E}u), u - \mathbb{E}u\rangle$. Every maximum over events below is restricted to $0 < \mu(B) < 1$. For a positive definite self-adjoint operator $L$ and a unit observation $v$, write

$$K_L(v) = \langle Lv, v\rangle\langle L^{-1}v, v\rangle.$$

For the normalized centered indicator $v_B$, this is the exact norm squared of binary conditioning in the energy seminorm, after constants have been removed. Direct diagonalization gives the identity

$$K_L(v) - 1 = \sum_{a<b} \frac{(\lambda_a - \lambda_b)^2}{\lambda_a \lambda_b} |\langle v, e_a\rangle|^2 |\langle v, e_b\rangle|^2, \tag{8}$$

where $(e_a)$ is an orthonormal eigenbasis. Every spectral cross pair is retained. The Kantorovich upper bound is classical; our first question is the optimal lower bound when only the specified event observations are available.

## 3.1 The exact three-atom constant

**Theorem 3.1** (Sharp unequal-atom stability)**.** *Let $p_1 \geq p_2 \geq p_3 > 0$, $p_1 + p_2 + p_3 = 1$, and let $L$ be positive definite and self-adjoint on the two-dimensional real mean-zero space of these three atoms. Put $\kappa = \lambda_{\max}(L)/\lambda_{\min}(L)$ and $K_{\max} = \max_B K_L(v_B)$. Then*

$$C(p)\frac{(\kappa - 1)^2}{\kappa} \leq K_{\max} - 1 \leq \frac{(\kappa - 1)^2}{4\kappa}, \qquad C(p) = \frac{p_3}{4(1 - p_1)(1 - p_2)}. \tag{9}$$

*Both constants are optimal for every fixed $\kappa > 1$ as the eigenbasis varies. Lower equality occurs exactly when the two orthogonal eigenlines bisect a closest pair of singleton observation lines. Upper equality occurs when a singleton line has angle $\pi/4$ to an eigenline. At $\kappa = 1$ every orientation gives equality.*

*If $\varepsilon \geq 0$ and $K_{\max} \leq 1 + \varepsilon$, then, with $c = (\lambda_{\max} + \lambda_{\min})/2$,*

$$(1 - \rho)c\,\mathrm{Var}(u) \leq q(u) \leq (1 + \rho)c\,\mathrm{Var}(u), \qquad \rho = \sqrt{\frac{\varepsilon}{\varepsilon + 4C(p)}}. \tag{10}$$

*This modulus is sharp on the fixed probability space.*

*Proof.* The three nontrivial binary projections are the singleton projections. Their unit centered directions satisfy

$$\langle v_i, v_j \rangle = -\sqrt{\frac{p_i p_j}{(1-p_i)(1-p_j)}} \quad (i \neq j), \qquad \sum_i \sqrt{p_i(1-p_i)}\, v_i = 0.$$

The positive relation and the negative pairwise inner products put the three oriented vectors in cyclic order, with successive angles between $\pi/2$ and $\pi$. Their unoriented lines therefore have three successive gaps

$$\alpha_{ij} = \arccos \sqrt{\frac{p_i p_j}{(1-p_i)(1-p_j)}} \in (0, \pi/2),$$

in a suitable order; these gaps sum to $\pi$.

If an observation line has angle $\theta$ to a lower eigenline, (8) becomes

$$K_L(v) - 1 = \frac{(\kappa-1)^2}{4\kappa} \sin^2(2\theta).$$

Double the three line angles modulo $\pi$. The new circular gaps are $\pi - 2\alpha_{ij}$ in a possibly different order. Their largest gap is $G = \pi - 2\alpha_{\min}$. On a circle of circumference $\pi$, the shortest arc containing a given finite set has length $\pi - G$: its complementary open arc is a largest gap. A ball of radius $r < \pi/2$ has arc length $2r$. Since $\sin^2$ increases with circular distance to zero on $[0, \pi/2]$, it follows that

$$\min_{\phi} \max_{i} \sin^2(2\theta_i - 2\phi) = \sin^2 \alpha_{\min}.$$

The minimizing centers are precisely the midpoints of the shortest containing arcs. Returning to the original projective circle identifies the asserted bisecting eigenlines.

Since $p/(1-p)$ increases, the minimum angle belongs to the pair $(p_1, p_2)$, and

$$\sin^2 \alpha_{12} = 1 - \frac{p_1 p_2}{(1-p_1)(1-p_2)} = \frac{p_3}{(1-p_1)(1-p_2)}.$$

This proves the lower bound and its equality cases. The upper bound and its attainability follow from $\sin^2 \leq 1$.

Writing $s = (\kappa-1)^2/\kappa$, the hypothesis gives $s \leq \varepsilon/C(p)$. The optimal relative spectral width is

$$\left(\frac{\lambda_{\max} - \lambda_{\min}}{\lambda_{\max} + \lambda_{\min}}\right)^2 = \frac{s}{s+4} \leq \frac{\varepsilon}{\varepsilon + 4C(p)}.$$

The lower-equality orientation, with $s = \varepsilon/C(p)$, proves sharpness. The case $\varepsilon = 0$ is the scalar operator. □

At equal probabilities $C(p) = 3/16$, recovering the original three-atom theorem exactly. If $p_{\min} \geq a$ with $0 < a \leq 1/3$, then $C(p) \geq a/(1+a)^2$, sharply at $((1-a)/2, (1-a)/2, a)$. Indeed, for fixed $p_3$, the denominator is maximized at $p_1 = p_2$, and $a/(1+a)^2$ increases for $a \leq 1/3$.

**Corollary 3.2** (Rare-event failure of uniform stability)**.** *There is no probability-space-independent relative variance modulus $\rho(\varepsilon) \to 0$ under uniformly $(1+\varepsilon)$-bounded binary conditioning, even on probability spaces with exactly three positive atoms and bounded positive definite energy operators.*

*Proof.* Take probabilities $((1-\delta)/2, (1-\delta)/2, \delta)$, $0 < \delta \le 1/3$. An orthonormal mean-zero basis is

$$e = (1, -1, 0)/\sqrt{1-\delta}, \qquad h = v_{\{3\}}.$$

For $\kappa \ge 1$, set $Le = e$, $Lh = \kappa h$. The third observation is an eigenvector; each of the other two has squared $h$-component $\delta/(1+\delta)$. Thus

$$K_{\max} - 1 = \frac{(\kappa-1)^2}{\kappa} \frac{\delta}{(1+\delta)^2}.$$

Keep any fixed $\kappa > 1$ and let $\delta \downarrow 0$. A relative comparison with width $\rho < 1$ would imply $\kappa \le (1+\rho)/(1-\rho)$, contradicting $\rho \to 0$. Taking $\kappa = \delta^{-1/2}$ even makes the condition number diverge while the maximal conditional cost tends to one. □

### 3.2 Dimension growth with equal probabilities

The harmonic profile below is the signed, centered version of the rank-one indicator-norm example in [16, Section 1.3]. The additional calculation is its exact binary-conditioning cost.

**Proposition 3.3** (An all-binary harmonic obstruction)**.** *On $2m$ equiprobable atoms, $m \ge 2$, put $H_m = \sum_{i=1}^{m} i^{-1}$ and define the unit centered vector*

$$e(i, \pm) = \pm\sqrt{m/H_m}\, i^{-1/2}.$$

*For $\kappa \ge 1$ and $L = I + (\kappa - 1)e \otimes e$ on the mean-zero space,*

$$K_{\max} - 1 \le \frac{4(\kappa-1)^2}{\kappa H_m}. \tag{11}$$

*Consequently a dimension-independent stability modulus fails even when every atom has equal probability and every binary observation is tested.*

*Proof.* The vector has mean zero and squared norm one. For $t_B = |\langle v_B, e\rangle|^2$, direct inversion of the rank-one perturbation gives

$$K_L(v_B) - 1 = \frac{(\kappa-1)^2}{\kappa} t_B(1 - t_B).$$

Replace $B$ by its complement so that $k = |B| \le m$. The absolute sum of any $k$ entries of $e$ is bounded by the sum of its $k$ largest positive entries. Since $\sum_{i=1}^{k} i^{-1/2} \le 2\sqrt{k}$,

$$|\mathbb{E}(e1_B)|^2 \le \frac{k}{mH_m}, \qquad \mu(B)(1-\mu(B)) \ge \frac{k}{4m}.$$

Since $v_B$ and $e$ are unit vectors, $0 \le t_B \le 1$; together with $t_B \le 4/H_m$ this proves (11). With fixed $\kappa > 1$, the right side tends to zero. For the varying choice $\kappa_m = H_m^{1/2}$, let $L_m = I + (\kappa_m - 1)e \otimes e$. Since $H_m \ge 1$ and $(\sqrt{H_m} - 1)^2 \le H_m$, the estimate is explicitly

$$0 \le K_{\max}(L_m) - 1 \le \frac{4(\sqrt{H_m}-1)^2}{H_m^{3/2}} \le \frac{4}{\sqrt{H_m}} \longrightarrow 0, \qquad \kappa_m \longrightarrow \infty. \tag{12}$$

Both limits follow from $H_m \to \infty$. The same spectral-endpoint argument as in Corollary 3.2 proves the failure of a uniform modulus. □

The estimate implies that the best possible condition-number lower constant on $2m$ equal atoms is at most $4/H_m$. A matching lower bound or optimal logarithmic asymptotic is not asserted. The constant 4 is, however, asymptotically sharp for the displayed harmonic family: the event of all positive atoms has $t_B = (\sum_{i=1}^m i^{-1/2})^2/(mH_m)$. Integral lower and upper bounds give $m^{-1/2}\sum_{i=1}^m i^{-1/2} \to 2$. In particular $t_B \to 0$ and $H_m t_B(1-t_B) \to 4$. Together with the uniform upper bound, this yields $H_m(K_{\max}-1)/((\kappa-1)^2/\kappa) \to 4$ for any choice $\kappa = \kappa_m > 1$.

Singleton testing can be much weaker than all binary testing. For a balanced sign vector $e$ on an even number $n$ of equal atoms, $|\langle v_i, e\rangle|^2 = 1/(n-1)$. The same rank-one perturbation therefore has singleton cost excess $[(\kappa-1)^2/\kappa](n-2)/(n-1)^2$. When 4 divides $n$, an event consisting of $n/4$ positive atoms has squared overlap $1/3$, hence cost excess $2(\kappa-1)^2/(9\kappa)$. This is an exact comparison for a specified family, not a universal lower estimate for all forms.

**Theorem 3.4** (An explicit lower bound for every finite probability vector)**.** *Let $n \geq 3$, let $p_i > 0$ have sum one, and let $L > 0$ be self-adjoint on the corresponding mean-zero space. Put $\kappa = \lambda_{\max}(L)/\lambda_{\min}(L) \geq 1$. Testing all binary observations gives*

$$K_{\max} - 1 \geq \frac{1}{4(\sum_i p_i^{-1} + n^2)} \frac{(\kappa-1)^2}{\kappa} \geq \frac{p_{\min}}{8n} \frac{(\kappa-1)^2}{\kappa}. \tag{13}$$

*In fact singleton and two-atom events suffice. These constants are not asserted to be optimal.*

*Proof.* For $\kappa = 1$ the assertion is immediate. Otherwise choose orthonormal minimum- and maximum-eigenvalue vectors $x, y$, and set $a_i = p_i x_i$, $b_i = p_i y_i$. Let $\mathcal{B}_2$ consist of the singleton and two-atom events, and define

$$\delta = \max_{B \in \mathcal{B}_2} |\langle v_B, x\rangle\langle v_B, y\rangle|.$$

Singletons imply $|a_i b_i| \leq \delta p_i(1-p_i) \leq \delta p_i$. For $i \neq j$, the pair event implies

$$|(a_i + a_j)(b_i + b_j)| \leq \delta(p_i + p_j)(1 - p_i - p_j) \leq \delta(p_i + p_j).$$

Subtracting the two diagonal terms proves $|a_i b_j + a_j b_i| \leq 2\delta(p_i + p_j)$. The same, possibly loose, estimate holds for $i = j$ by the singleton bound. Normalization and orthogonality give the exact identity

$$\sum_{i,j} \frac{(a_i b_j + a_j b_i)^2}{p_i p_j} = 2.$$

Therefore

$$2 \leq 4\delta^2 \sum_{i,j} \frac{(p_i + p_j)^2}{p_i p_j} = 8\delta^2 \left( \sum_i p_i^{-1} + n^2 \right).$$

The nonnegative sum in (8) includes the minimum/maximum eigenvalue pair, yielding $K_{\max} - 1 \geq ((\kappa-1)^2/\kappa)\delta^2$. Finally $\sum_i p_i^{-1} \leq n/p_{\min}$ and $n^2 \leq n/p_{\min}$ prove the second estimate. □

Singletons alone cannot give a positive coefficient of $(\kappa-1)^2/\kappa$ uniformly over all energies, even on four equal atoms. Take the orthonormal mean-zero vectors

$$x = (\sqrt{2}, -\sqrt{2}, 0, 0), \quad y = (0, 0, \sqrt{2}, -\sqrt{2}), \quad z = (1, 1, -1, -1),$$

with eigenvalues $\kappa^{-1/2}, \kappa^{1/2}, 1$ respectively, where $\kappa > 1$. Each singleton has spectral masses $2/3, 1/3$ on adjacent spectral values, so its cost excess is $\frac{2}{9}(\sqrt{\kappa} + \kappa^{-1/2} - 2)$. The ratio to $(\kappa-1)^2/\kappa$ tends to zero as $\kappa \to \infty$. The event $\{1, 3\}$ instead has direction $(x+y)/\sqrt{2}$ and cost excess $(\kappa-1)^2/(4\kappa)$. This distinction does not contradict the logarithmic singleton estimate below.

### 3.3 Quantitative observation connectivity

For a finite nonempty family of unit observations $(v_j)_{j=1}^M$ in a real $d$-dimensional space, $d \geq 2$, write $Q_j = v_j \otimes v_j$, choose weights $\omega_j > 0$, and put $W = \sum_j \omega_j$. Define the observation commutator gap

$$\gamma = \min_{\substack{S=S^*,\ \operatorname{tr} S=0 \\ \|S\|_{\mathrm{HS}}=1}} \sum_j \omega_j \|[S, Q_j]\|_{\mathrm{HS}}^2. \tag{14}$$

It is an eigenvalue of an explicitly specified finite quadratic form. In particular it is independently checkable without knowing the energy operator $L$.

**Theorem 3.5** (Conditional cost and logarithmic scalarity)**.** *Suppose $\gamma > 0$, $\varepsilon \geq 0$, and $L > 0$ is self-adjoint. If $K_L(v_j) \leq 1 + \varepsilon$ for every $j$, then*

$$\log \frac{\lambda_{\max}(L)}{\lambda_{\min}(L)} \leq 2\sqrt{\frac{\varepsilon W}{\gamma}}. \tag{15}$$

*With $c = (\lambda_{\max} + \lambda_{\min})/2$, the corresponding relative scalar comparison has coefficient $\rho = \tanh \sqrt{\varepsilon W/\gamma}$.*

*If the observations span the space and their graph of nonzero inner products is connected, then $\gamma > 0$.*

*Proof.* Set $S = \log L$ and diagonalize $L$. Formula (8) and $2|\sinh(t/2)| \geq |t|$ give

$$K_L(v) - 1 = 4 \sum_{a<b} \sinh^2\left(\frac{\log \lambda_a - \log \lambda_b}{2}\right) x_a^2 x_b^2 \geq \frac{1}{2}\|[S, Q_v]\|_{\mathrm{HS}}^2.$$

Thus the definition of $\gamma$ yields

$$\left\| S - \frac{\operatorname{tr} S}{d} I \right\|_{\mathrm{HS}}^2 \leq \frac{2\varepsilon W}{\gamma}.$$

The squared difference of any two eigenvalues is at most twice the sum of their squared deviations from their mean. Applying this to $S$ proves (15). The relative width identity $(M-m)/(M+m) = \tanh[\log(M/m)/2]$ gives the comparison.

Finally a self-adjoint operator commuting with $Q_j$ has $v_j$ as an eigenvector. Nonzero inner products force equality of the corresponding eigenvalues. Connectivity and spanning therefore make the common commutant scalar. No nonzero traceless symmetric operator annuls the quadratic form in (14); compactness of its unit sphere proves strict positivity of $\gamma$. □

**Proposition 3.6** (Exact simplex gap)**.** *For the $n \geq 3$ singleton observations on $n$ equiprobable atoms, with weights one,*

$$\gamma = \frac{2n}{(n-1)^2}.$$

*Thus singleton conditional costs at most $1 + \varepsilon$ imply $\rho \leq \tanh((n-1)\sqrt{\varepsilon/2})$ for every nonzero nonnegative energy form on the entire mean-zero space. This stability bound is not claimed sharp.*

*Proof.* Put $d = n - 1$. The regular simplex identities are $\sum_j Q_j = (n/d)I$ and $\langle v_i, v_j \rangle = -1/d$ for $i \neq j$. For traceless symmetric $S$,

$$\sum_j \|[S, Q_j]\|_{\mathrm{HS}}^2 = \frac{2n}{d}\|S\|_{\mathrm{HS}}^2 - 2\sum_j \langle Sv_j, v_j \rangle^2.$$

The matrices $R_j = Q_j - I/d$ have Gram diagonal $(d-1)/d$ and constant off-diagonal $-(d-1)/d^2$. Their nonzero Gram eigenvalue is $(d^2-1)/d^2$, so this is the squared norm of $S \mapsto (\langle S, R_j\rangle)_j$. It is attained on the span of the $R_j$. Substitution gives $\gamma = 2n/d - 2(d^2-1)/d^2 = 2n/d^2$.

For a nonnegative form, a zero-energy vector stays zero-energy under every finite-cost observation. Since every pair of singleton indicators has nonzero inner product, either all singleton energies vanish, making the form zero, or all are positive. In the latter case a zero-energy vector is orthogonal to every singleton and hence zero. The form is positive definite on the mean-zero space, so Theorem 3.5 applies. □

For any weighted rank-one family in dimension $d \geq 2$, one has $\gamma/W \leq 2/(d+2)$. Indeed the space of traceless real symmetric matrices has dimension $(d-1)(d+2)/2$. The trace of the quadratic form $S \mapsto \|[S,Q]\|_{\mathrm{HS}}^2$ for one rank-one projection is $d-1$: in a basis making $Q$ the first coordinate projection, exactly the $d-1$ normalized symmetric off-diagonal matrices joining that coordinate contribute one. The smallest eigenvalue of the weighted sum cannot exceed its average eigenvalue. Thus this particular normalized Hilbert–Schmidt gap cannot remain uniformly positive through refinements of unbounded dimension.

**Proposition 3.7** (Optimal normalized gap for four equal atoms)**.** *For the seven binary observation lines on four equal atoms, give each singleton line weight $b > 0$ and each of the three distinct pair lines weight $a > 0$. Then*

$$\gamma = \min\{8b/3,\ 2a + 8b/9\}, \qquad W = 4b + 3a.$$

*The choice $a = 8b/9$ attains $\gamma/W = 2/5$, the largest possible normalized Hilbert–Schmidt gap for any weighted rank-one observation family in dimension three. The resulting conditional-cost estimate is $\rho \leq \tanh\sqrt{5\varepsilon/2}$; this last nonlinear estimate is not claimed to be sharp.*

*Proof.* Identify the mean-zero space with $\mathbb{R}^3$. Its singleton lines are the four tetrahedral directions $(1,1,1)/\sqrt{3}$, $(1,-1,-1)/\sqrt{3}$, $(-1,1,-1)/\sqrt{3}$, and $(-1,-1,1)/\sqrt{3}$. The pair lines are the three coordinate axes. Diagonal traceless symmetric matrices have zero quadratic pairing with each tetrahedral direction, so the tetrahedral commutator form has eigenvalue $8b/3$ there. The regular-simplex computation above gives eigenvalue $8b/9$ on the three-dimensional off-diagonal symmetric subspace. The coordinate-axis projections contribute zero on the diagonal subspace and $2a$ on each normalized symmetric off-diagonal basis matrix. The two orthogonal subspaces exhaust the five-dimensional traceless symmetric space, proving the formula. The stated weight ratio makes both eigenvalues $8b/3$ and gives $W = 20b/3$. It attains the preceding upper bound $2/(d+2) = 2/5$. □

The geometry and the weight ratio in this proposition are established: after separating antipodal points, they are the weighted cube and cross-polytope spherical 5-design, with point weights $3/40$ and $1/15$ respectively; see [6, Example 2.6]. The present calculation applies that known geometry to binary energy costs. It does not claim a new cubature design.

The interpretation is precise. The author-developed ALH viewpoint retains the observation directions and their commutation defects; the displayed analytic gap must still be proved or computed. The SPADES covariance decomposition identifies their actual overlaps and keeps the cross-energy terms. FINER can index simultaneous observation and regularization refinements, but does not manufacture uniform geometric coercivity. The sharp constants and obstructions above are analytic statements, distinct from exact germ identities or asymptotic resolution notation.

# 4 An arbitrarily small binary defect does not control the energy

All Hilbert spaces and forms in this section are real. For a probability space, a binary observation is conditional expectation onto the sigma algebra generated by one event. Exact contractivity under all such observations forces a closed dense energy vanishing on constants to be a multiple of variance, by Theorem 2.1. The assertion has no uniform approximate analogue, even at the level of boundedness of the energy operator.

**Theorem 4.1** (Infinite atomic obstruction)**.** *For every $\varepsilon > 0$ there are a probability measure $\mu$ on $\mathbb{N}_0$ and a densely defined closed nonnegative quadratic form $q$ on $H = L^2(\mu)$ with the following properties:*

1. *$1 \in \operatorname{Dom} q$, $q(1) = 0$, and the zero-energy subspace consists exactly of the constant functions.*
2. *Every indicator belongs to $\operatorname{Dom} q$. For every nontrivial event $B$, the binary conditional expectation $P_B$ preserves this domain and $q(P_B u) \le (1+\varepsilon) q(u)$ for all $u \in \operatorname{Dom} q$.*
3. *The form domain is a proper dense subspace of $H$. On the mean-zero subspace its Rayleigh quotients have infimum zero and supremum infinity. In particular its associated nonnegative selfadjoint operator is unbounded, injective on that subspace, and has zero in its spectrum.*

*Thus no conclusion of full form domain, bounded operator, positive spectral lower bound, or comparison with a finite scalar multiple of variance follows from an arbitrarily small positive uniform binary energy defect. This includes comparisons with constants depending on the particular probability space and form.*

We prove the theorem through an exact formula that keeps the energy of the complementary event.

**Lemma 4.2** (Atomic star form and exact binary cost)**.** *Let $p_0 > 0$, $p_n > 0$ for $n \ge 1$, and $\sum_{n\ge 0} p_n = 1$. Let $w_n > 0$ and suppose*

$$W = \sum_{n\ge 1} p_n w_n < \infty, \qquad V = \sum_{n\ge 1} p_n / w_n < \infty.$$

*On $H = \ell^2(p)$ define*

$$\operatorname{Dom} q = \left\{ f \in H : \sum_{n\ge 1} p_n w_n (f_n - f_0)^2 < \infty \right\}, \qquad q(f) = \sum_{n\ge 1} p_n w_n (f_n - f_0)^2.$$

*This is a densely defined closed nonnegative form, with kernel the constants, and every indicator is in its domain. If $\varnothing \ne B \subseteq \mathbb{N}$, set*

$$b = \sum_{n\in B} p_n, \quad W_B = \sum_{n\in B} p_n w_n, \quad V_B = \sum_{n\in B} p_n / w_n, \quad V_{B^c} = \sum_{n\in \mathbb{N}\setminus B} p_n / w_n.$$

*Then $0 < b < 1$ and the exact optimal binary constant is*

$$\sup_{f\in \operatorname{Dom} q,\ q(f)>0} \frac{q(P_B f)}{q(f)} = \frac{W_B V_B}{b^2} + \frac{W_B V_{B^c}}{(1-b)^2}. \tag{16}$$

*Events containing atom zero are covered by replacing them by their complements, which give the same binary projection.*

*Proof.* The linear map

$$Df = (\sqrt{p_n w_n}(f_n - f_0))_{n\geq 1}$$

from its maximal domain in $H$ to $\ell^2$ is closed. Indeed, convergence in $H$ implies convergence at every atom, including atom zero. If also $Df^{(k)} \to y$ in $\ell^2$, each coordinate of $y$ therefore equals the displayed expression for the limiting $f$. Consequently $f$ belongs to the domain and $Df = y$. The graph norm $\|f\|^2 + q(f)$ is complete. Finite-support functions belong to the domain: the only extra case is a nonzero value at atom zero, whose energy is controlled by $W$. They are dense in $H$. The kernel assertion follows from $p_n w_n > 0$, and indicators have energy at most $W$. In particular $P_B f$, being a linear combination of 1 and $1_B$, always belongs to the domain.

Put $x_n = f_n - f_0$. The exact centered observation functional is

$$\mathrm{Cov}(f, 1_B) = \sum_{n\geq 1} p_n x_n (1_B(n) - b).$$

These sums converge by Cauchy–Schwarz in $L^2(p)$, or by the weighted estimate below on the form domain. Since $q(1_B - b) = W_B$ and $\mathrm{Var}(1_B) = b(1-b)$,

$$q(P_B f) = \frac{W_B}{b^2(1-b)^2} |\mathrm{Cov}(f, 1_B)|^2.$$

The squared dual norm of the covariance functional on the energy coordinates $\ell^2(p_n w_n)$ is

$$\sum_{n\geq 1} \frac{p_n}{w_n} (1_B(n) - b)^2 = (1-b)^2 V_B + b^2 V_{B^c}.$$

Weighted Cauchy–Schwarz gives the upper bound. Finite-coordinate truncations of $x_n = (1_B(n) - b)/w_n$, with $f_0 = 0$, are admissible finite-support tests and approach that dual norm, by convergence of the nonnegative series. This proves optimality and (16). No complementary contribution is omitted. □

*Proof of Theorem 4.1.* Set $r = \min\{1/8, \varepsilon/3\} > 0$. For $n \geq 1$, take

$$p_n = r^n 2^{-2n^2}, \qquad w_n = 2^{(-1)^n n}, \qquad p_0 = 1 - \sum_{n\geq 1} p_n.$$

The successive ratios of each of the three positive sequences $p_n$, $p_n w_n$, and $p_n/w_n$ are at most $r$. For the latter two, this follows from

$$|(-1)^{n+1}(n+1) - (-1)^n n| = 2n+1, \qquad 2((n+1)^2 - n^2) = 4n+2.$$

In fact their ratios are at most $r2^{-(2n+1)}$. Since $p_1 = r/4$, the mass away from zero is at most $r/[4(1-r)] < 1$, so $p_0 > 0$. Both $W$ and $V$ are finite.

For any nonempty $B \subseteq \mathbb{N}$, let $m = \min B$. The geometric tail bounds give

$$W_B \leq \frac{p_m w_m}{1-r}, \qquad V_B \leq \frac{p_m/w_m}{1-r}, \qquad b \geq p_m.$$

Also $WV \leq p_1^2/(1-r)^2$ and $1 - b \geq 1 - r/[4(1-r)]$. Thus (16) gives, uniformly in every event,

$$K(B) \leq \frac{1}{(1-r)^2} + \frac{r^2}{(4-5r)^2} \leq 1 + 3r \leq 1 + \varepsilon. \tag{17}$$

For the second inequality, use $r \leq 1/8$: $(1-r)^{-2} - 1 \leq 128r/49$ and $r^2/(4-5r)^2 \leq r^2 \leq r/8$; their sum is strictly less than $3r$. This proves the required uniform projection estimate.

The normalized centered singleton indicators have Rayleigh quotients

$$\frac{q(1_{\{n\}} - p_n)}{\|1_{\{n\}} - p_n\|^2} = \frac{w_n}{1 - p_n}.$$

Along odd indices these tend to zero; along even indices they tend to infinity. To see directly that the form domain is proper, put $f_0 = 0$, $f_n = (p_n w_n)^{-1/2}$ for even $n$, and $f_n = 0$ for odd $n$. Then $\|f\|^2 = \sum_{n \text{ even}} 2^{-n} < \infty$, whereas $q(f) = \sum_{n \text{ even}} 1 = \infty$. The lemma proves all remaining closedness, density and kernel claims. The spectral assertions follow from the representation of a closed nonnegative form: a positive spectral lower bound or bounded upper spectrum would bound the corresponding Rayleigh quotients. The kernel contains only constants, so zero is not an eigenvalue on the mean-zero subspace. □

The example is also a symmetric Dirichlet form: normal contractions $\phi$, with $\phi(0) = 0$, satisfy $|\phi(f_n) - \phi(f_0)| \le |f_n - f_0|$, and hence $q(\phi \circ f) \le q(f)$. Thus adding the usual Markov contraction property does not repair the proposed approximate rigidity theorem. No infinitesimal construction or limiting interchange is needed for this counterexample; it occurs on one fixed countable probability space for each prescribed positive tolerance.

**Proposition 4.3** (Kernel alternative under finite binary costs)**.** *Let $q$ be a densely defined closed nonnegative form on real $L^2(\mu)$, with $1 \in \operatorname{Dom} q$ and $q(1) = 0$. Suppose every binary projection preserves the form domain and has some finite energy bound, allowed to depend on the event. Then either $q = 0$ on all of $L^2(\mu)$, or its kernel consists exactly of constants. This gives pointwise positivity on nonzero mean-zero domain vectors, not a positive uniform spectral lower bound.*

*Proof.* The form kernel $N$ intersected with the mean-zero subspace is closed in the state norm, by lower semicontinuity. Every binary projection preserves $N$, since its energy bound maps zero energy to zero energy. On the mean-zero space it is a rank-one orthogonal projection onto a normalized centered indicator $v_B$. If $0 \ne h \in N$, density of simple functions gives an event $B$ with $\langle h, v_B \rangle \ne 0$; hence $v_B \in N$. Whenever two centered indicators have nonzero covariance, membership of the first in $N$ implies membership of the second. If their covariance is zero, their intersection is nontrivial and has strictly positive covariance with both. Thus every centered indicator belongs to $N$, and density and closedness give $N = H \ominus \mathbb{R}1$. Including constants then makes the entire space the zero-energy domain. If no nonzero $h$ exists, the kernel is precisely the constants. □

## 5 Failure on a fixed nonatomic probability space

The preceding atomic construction is not an artifact of allowing the probability space to vary. On Lebesgue probability space, small normalized indicator tests can be combined into a single unbounded energy with arbitrarily small uniform binary defect. The basic observation is a logarithmic separation between these tests and the Hilbert norm.

**Lemma 5.1** (Small indicator tests on a nonatomic block)**.** *For every $m \ge 1$, write $H_m = \sum_{k=1}^{m} k^{-1}$. There is a mean-zero unit vector $h_m \in L^2(0,1)$, constant on $2m$ equal intervals, such that for every measurable event $B$,*

$$\left| \int_B h_m \right|^2 \le \frac{4}{H_m} |B|(1 - |B|). \tag{18}$$

*If this vector is rescaled to have unit norm on any positive-length interval $I \subset (0,1)$ and extended by zero, the resulting mean-zero unit vector $e$ satisfies, for every nontrivial measurable*

$B$,

$$\frac{|\langle 1_B - |B|, e\rangle|^2}{|B|(1-|B|)} \le \frac{8}{H_m}. \tag{19}$$

*These bounds apply to all measurable events, including events that cut the intervals on which the vector is constant.*

*Proof.* Assign the values $\pm\sqrt{m/H_m}\,k^{-1/2}$, for $1 \le k \le m$, to the $2m$ equal intervals. The paired signs prove zero mean, and the sum of their squared integrals equals one. Replace $B$ by its complement if necessary, so $b = |B| \le 1/2$; the integral changes sign because $h_m$ has mean zero. For fixed $b$, the largest possible positive integral is obtained by filling positive intervals in decreasing order of their values, with at most one interval only partly filled. This follows directly by transferring mass from a smaller value to a larger one. Apply the same argument to $-h_m$ for the absolute value.

Put $k = 2mb \in [0, m]$. The sum of the first $\lfloor k \rfloor$ values $j^{-1/2}$ plus the appropriate fraction of the next one is at most $2\sqrt{k}$. At integer $k$, use $\sum_{j=1}^{k} j^{-1/2} \le \int_0^k x^{-1/2}\,dx$. Between consecutive integers, the left side is its linear interpolant, and concavity of $2\sqrt{k}$ bounds that interpolant by the same function. Consequently

$$\left|\int_B h_m\right| \le \frac{1}{2m}\sqrt{\frac{m}{H_m}}\,2\sqrt{k}, \qquad \left|\int_B h_m\right|^2 \le \frac{2b}{H_m} \le \frac{4}{H_m}b(1-b).$$

The endpoint $b = 0$ is immediate. This proves the first assertion.

For a block $I$ of length $a$, write $s = |B \cap I|$. Rescaling the first inequality gives $|\int_B e|^2 \le (4/H_m)s(1 - s/a) \le 4s/H_m$. Again replace $B$ by its complement so that $b \le 1/2$. Since $s \le b$ and $b(1-b) \ge b/2$, division yields (19). □

**Theorem 5.2** (Nonatomic obstruction at every positive tolerance)**.** *For every $\varepsilon > 0$ there is a densely defined closed nonnegative form $q$ on the fixed real space $L^2(0,1)$ such that:*

1. *its kernel is exactly the constant functions;*
2. *every binary conditional expectation preserves its domain and $q(P_B u) \le (1+\varepsilon)q(u)$ for every measurable nontrivial $B$ and every $u \in \operatorname{Dom} q$;*
3. *the form domain is proper, and the mean-zero Rayleigh quotients have infimum zero and supremum infinity.*

*Every centered indicator also belongs to the domain of the inverse-square-root energy on the mean-zero subspace. Thus the obstruction persists even when both the forward and dual binary tests are individually finite.*

*Proof.* Set $\delta = \min\{1/2, \varepsilon/3\} > 0$. Choose pairwise disjoint intervals $I_j \subset (0,1)$ of positive length, for $j \ge 1$. Since $H_m \to \infty$, choose finite integers $m_j$ with $H_{m_j} \ge 8\,4^j/\delta$. Lemma 5.1 provides mean-zero unit vectors $e_j$ supported in $I_j$ with

$$t_j(B) := |\langle v_B, e_j\rangle|^2 \le \delta\,4^{-j}, \qquad v_B = \frac{1_B - |B|}{\sqrt{|B|(1-|B|)}}.$$

Disjoint supports make the $e_j$ orthonormal. On $H_0 = L^2(0,1) \ominus \mathbb{R}1$, let $E = \overline{\operatorname{span}}\{e_j : j \ge 1\}$ and decompose $u = u_\perp + \sum_j a_j e_j$ with $u_\perp \in H_0 \ominus E$. Take $\lambda_j = 2^{(-1)^j j}$ and define the positive diagonal selfadjoint operator $L$ by $Le_j = \lambda_j e_j$ and $L = I$ on $H_0 \ominus E$, with operator domain $\{u \in H_0 : \sum_j \lambda_j^2 |a_j|^2 < \infty\}$. Its form is

$$q(u) = \|u_\perp\|^2 + \sum_j \lambda_j |a_j|^2, \quad \operatorname{Dom} q = \left\{ u : \sum_j \lambda_j |a_j|^2 < \infty \right\}.$$

Extend it to $H$ by assigning zero energy to constants. The spectral diagonal construction, or completeness of the displayed weighted coordinate norm, proves closedness and density. Every coefficient $\lambda_j$ is positive, so the kernel assertion holds.

For a normalized centered indicator, orthogonality and $\sum_j t_j(B) \le 1$ imply

$$q(v_B) = 1 - \sum_j t_j(B) + \sum_j \lambda_j t_j(B) \le 1 + \delta,$$
$$\|L^{-1/2} v_B\|^2 = 1 - \sum_j t_j(B) + \sum_j \lambda_j^{-1} t_j(B) \le 1 + \delta,$$

because both $\lambda_j$ and $\lambda_j^{-1}$ are at most $2^j$ and $\sum_j 2^j \delta 4^{-j} = \delta$. These estimates establish membership in both domains; they are not formal expressions involving an infinite undefined energy. Since $P_B u$ is a linear combination of 1 and $v_B$, domain invariance follows. For $u \in \operatorname{Dom} q$, the exact weighted Hilbert pairing and Cauchy–Schwarz give

$$|\langle u - \mathbb{E}u, v_B \rangle|^2 \le q(u)\, \|L^{-1/2} v_B\|^2.$$

It follows that $q(P_B u) \le q(v_B) \|L^{-1/2} v_B\|^2 q(u) \le (1+\delta)^2 q(u) \le (1+\varepsilon) q(u)$. The last inequality uses $2\delta + \delta^2 \le 3\delta \le \varepsilon$.

The vectors $e_j$ themselves have Rayleigh quotients $\lambda_j$, which tend to zero along odd indices and to infinity along even indices. The vector $u = \sum_{j \text{ even}} 2^{-j/2} e_j$ lies in $H_0$ but has infinite form energy. This proves properness of the domain directly. □

Neither theorem asserts that the energy defect controls every orthogonal rank-one projection. That stronger test family would include all mixtures of spectral directions, and the Kantorovich extremal formula would control the spectral condition number. Centered indicators form a total family, but totality does not give a uniform norming estimate in an infinite-dimensional Hilbert space. The construction exploits this distinction quantitatively. Its spectral form need not be Markovian; the atomic construction above provides a separate Markovian example.

## 6 Filtered infinitesimal nets in standard analysis

All spaces are real unless stated otherwise. A proper filter $\mathcal{F}$ on an index set $I$ specifies the sets of indices on which a statement holds eventually. A directed set supplies its tail filter. A positive scale $\varepsilon : I \to (0, \infty)$ is infinitesimal when $\varepsilon \to 0$ along this filter. These are ordinary real numbers at every index. The word "infinitesimal" describes their limiting role.

There are two distinct constructions. The exact germ ring

$$\mathcal{G}_{\mathcal{F}} = \mathbb{R}^I / \sim, \qquad x \sim y \iff \{i : x_i = y_i\} \in \mathcal{F},$$

has pointwise algebraic operations and an injective constant embedding of $\mathbb{R}$. A strictly positive vanishing scale defines a nonzero element $[\varepsilon]$ in this ring. It is an actual algebraic element, but not an infinitesimal member of the real field: the ring generally has zero divisors and only a partial eventual order. For example, the even and odd indicator sequences multiply to zero in the tail germ ring on $\mathbb{N}$. This Peano-indexed exact layer is the classical Schmieden–Laugwitz construction [24]; general filter indexing belongs to ordinary net theory [21].

An optional resolution quotient retains polynomial-scale errors. Set $\mathcal{M}_\varepsilon = \{x : |x_i| = O(\varepsilon_i^{-k}) \text{ for some } k \in \mathbb{N}\}$ and $\mathcal{N}_\varepsilon = \{x : |x_i| = O(\varepsilon_i^k) \text{ for every } k \in \mathbb{N}\}$. The quotient $\mathcal{M}_\varepsilon / \mathcal{N}_\varepsilon$ is an asymptotic algebra of the established generalized-function type [8, 10]. Refinement maps preserving eventual estimates and the induced algebra constructions are also established [19].

The germ of $e^{-1/\varepsilon}$ is nonzero in the exact layer and zero at this polynomial resolution. A convergent germ has a unique real standard part; an oscillating germ need not have one. No ultrafilter, transfer principle, or new axiom of nonstandard analysis [23] is used. Hardy-field restrictions of regular germs are optional and play no role in the proofs below.

For a closed linear response $T$, equivalence of nets in $\operatorname{Dom} T$ must retain its graph:

$$(u_i) \simeq_T (v_i) \quad \Longleftrightarrow \quad \|u_i - v_i\| + \|T(u_i - v_i)\| \longrightarrow 0.$$

State equivalence alone cannot descend the energy. On $\ell^2$, let $Te_n = ne_n$ and $u_n = e_n/n$. Then $u_n \to 0$ whereas $\|Tu_n\|^2 = 1$. This is a concrete loss of information, independent of any name for the indexing system. Graph equivalence preserves squared energies on graph-bounded families, in particular on graph-convergent recovery nets, by $|\|Tu_i\|^2 - \|Tv_i\|^2| \le (\|Tu_i\| + \|Tv_i\|)\|T(u_i - v_i)\|$. Without boundedness it need not do so: for $T = I$, $u_n = n$ and $v_n = n + n^{-1}$ are graph equivalent but their squared energies differ by a quantity tending to 2.

**Definition 6.1** (FINER data)**.** Filtered Infinitesimal Nets for Energetic Recovery (FINER) consist here of a proper filtered index carrying vanishing scales, a specified state Hilbert space, closed finite-stage energies, and identified response maps into one Hilbert space. Conditional projections and weights are additional specified data. Recovery means convergence of the identified states and responses together, with the asserted energies. This definition asserts no existence or compatibility. Theorem 7.2 constructs such nets from a fixed closed energy and arbitrary specified conditional projections.

This retains the original FINER distinction between exact germs, resolution quotients, and analytic realization. We use only the exact and graph layers because no polynomial rate for a general closed form is available. Germ notation does not prove any estimate. Conversely, the conditional response is not discarded merely because its state norm has a limit.

**Lemma 6.2** (Completeness and refinement)**.** *If $T : \operatorname{Dom} T \subset H \to Y$ is closed and linear and $u_n \in \operatorname{Dom} T$ satisfy*

$$\left(\|u_{n+1} - u_n\|^2 + \|T(u_{n+1} - u_n)\|^2\right)^{1/2} \le b_n, \qquad \sum_n b_n < \infty,$$

*then $u_n$ converges in the graph norm to $u \in \operatorname{Dom} T$, with error at most $\sum_{k \ge n} b_k$. For arbitrary filtered nets the graph Cauchy condition gives the same existence conclusion, without a series representation.*

*Proof.* The graph of $T$ is a closed subspace of the complete space $H \oplus Y$. The triangle inequality bounds each finite telescoping sum; its tails are Cauchy and give the asserted limit and error. Completeness of a metric uniform space also applies to every Cauchy filter. □

Thus natural-number recursion supplies an index when countable cofinality is available; Cantor completeness supplies a limit only after the correct graph-Cauchy estimate. Nested nonempty closed sets with shrinking graph diameters give the equivalent completeness argument. Neither principle supplies an energetic estimate on its own. Banach closed-graph and Riesz methods are used explicitly below; Hilbert geometry is essential to the exact squared-error identities.

## 6.1 Conditional information and cross terms

For $X, Y \in L^2(\mu)$ and a conditional expectation $P$, orthogonality gives, with $CX = X - \mathbb{E}X$,

$$\langle CX, CY \rangle = \langle PCX, PCY \rangle + \langle (I - P)X, (I - P)Y \rangle.$$

The first term is recoverable information and the second is the mean conditional covariance. The identity follows by expanding and using $P = P^* = P^2$; no independence hypothesis is used. It implies $\operatorname{Var} X = \operatorname{Var}(PX) + \mathbb{E}\operatorname{Var}(X \mid \mathcal{G})$. For a derivative energy $a$, the additional cross terms $a(PX, (I-P)Y)$ and $a((I-P)X, PY)$ generally remain. They vanish only under an independently justified reducing property. This distinction is the interface between conditional information and variational recovery used in this article.

# 7 Conditional energetic completion and recovery

This section gives the central constructive result. Its closed-form limit is an instance of monotone convergence of quadratic forms. The additional object retained here is the entire family of specified conditional responses in one Hilbert direct sum. The domain and the response vectors are part of the conclusion, rather than being inferred from convergence of a single scalar energy.

Let $H$ be a real Hilbert space, let $A \geq 0$ be self-adjoint, and put $V = \operatorname{Dom} A^{1/2}$ and $q(u) = \|A^{1/2}u\|^2$. Let $(P_j)_{j\in J}$ be any family of orthogonal projections on $H$, and let $w_j > 0$. Neither countability of $J$ nor summability of the weights is assumed. The principal probabilistic instance is $H = L^2(\Omega, \mu; \mathbb{R})$ and $P_j = \mathbb{E}_\mu[\,\cdot \mid \mathcal{G}_j]$; all projections then act on the same probability space. The $\mathcal{G}_j$ need not be nested. No assertion about changing reference measures is implicit.

Write

$$\mathcal{Y} = H \oplus \bigoplus_{j\in J} H, \qquad B_t = A^{1/2}(I + tA)^{-1/2} \quad (t > 0).$$

The spectral theorem gives a bounded operator $B_t$ with $\|B_t\| \leq t^{-1/2}$. For a finite $F \subset J$ define

$$D_{F,t}u = \big(A^{1/2}u, (1_F(j)\sqrt{w_j}B_tP_ju)_{j\in J}\big), \tag{20}$$

$$e_{F,t}(u) = \|D_{F,t}u\|_{\mathcal{Y}}^2 = q(u) + \sum_{j\in F} w_j\|B_tP_ju\|^2, \qquad u \in V. \tag{21}$$

Every $e_{F,t}$ is a densely defined closed form: its extra summands are bounded state forms. The order $(F,t) \preceq (F',t')$ means $F \subset F'$ and $t' \leq t$. It makes this set directed, and $e_{F,t} \leq e_{F',t'}$ on $V$.

**Theorem 7.1** (Conditional-energy completion and response representation)**.** *The extended quadratic form*

$$Q(u) = q(u) + \sum_{j\in J} w_jq(P_ju) = \sup_{F,t} e_{F,t}(u) \tag{22}$$

*is closed on $H$, with domain*

$$V_* = \left\{ u \in V : P_ju \in V \ (j \in J), \quad \sum_{j\in J} w_jq(P_ju) < \infty \right\}.$$

*A sum of nonnegative terms over $J$ denotes the supremum of its finite subsums, and an undefined form value means $+\infty$. On this domain the operator*

$$Du = \big(A^{1/2}u, (\sqrt{w_j}A^{1/2}P_ju)_{j\in J}\big) : V_* \subset H \longrightarrow \mathcal{Y}$$

*is closed and $Q(u) = \|Du\|^2$. The domain $V_*$ need not be dense. On $K = \overline{V_*}^{\,H}$, $Q$ is densely defined and determines a nonnegative self-adjoint operator $A_*$.*

*Proof.* For any $v \in H$, monotone convergence for its spectral measure gives

$$\sup_{t>0} \|B_t v\|^2 = \int_{[0,\infty)} \lambda \, d\|E_A(\lambda)v\|^2 = q(v),$$

including the value $+\infty$. This is monotone convergence of an ordinary scalar integral, not an interchange of an integral with an arbitrary net. Taking finite subsums proves (22). Each extended $e_{F,t}$ is lower semicontinuous, so their supremum is lower semicontinuous. Alternatively, if $u_n \to u$ in $H$ and $Du_n \to y$ in $\mathcal{Y}$, then closedness of $A^{1/2}$ gives $u \in V$ and $y_0 = A^{1/2}u$. Since $P_j u_n \to P_j u$ in $H$, the same argument in each coordinate gives $P_j u \in V$ and $y_j = \sqrt{w_j} A^{1/2} P_j u$. Membership of $y$ in the direct sum proves $u \in V_*$ and $Du = y$. This proves closedness of $D$, and hence completeness of the $Q$ graph norm. Restriction to $K$ permits the usual closed-form representation theorem. □

**Theorem 7.2** (Simultaneous filtered energetic recovery)**.** *Fix $u \in V_*$. For every $(F, t, z)$ with $F \subset J$ finite and $t, z > 0$, let $u_{F,t,z} \in V$ be the unique minimizer of*

$$v \longmapsto e_{F,t}(v) + z\|v - u\|^2. \tag{23}$$

*Equivalently, $u_{F,t,z} = z(A_{F,t} + z)^{-1}u$, where $A_{F,t}$ is associated with $e_{F,t}$. Along the directed product $F \uparrow J$, $t \downarrow 0$, $z \uparrow \infty$,*

$$\|u_{F,t,z} - u\|^2 \le Q(u)/z, \tag{24}$$

$$D_{F,t}u_{F,t,z} \longrightarrow Du \quad \textit{strongly in } \mathcal{Y}. \tag{25}$$

*Consequently $e_{F,t}(u_{F,t,z}) \to Q(u)$, and all specified conditional responses are recovered simultaneously by the regularized finite-stage responses.*

*The same conclusions hold for approximate minimizers $v_{F,t,z}$ whose objective in (23) is at most its minimum plus $\delta_{F,t,z} \ge 0$, provided $\delta_{F,t,z} \to 0$. More precisely,*

$$\|D_{F,t}(v_{F,t,z} - u_{F,t,z})\|^2 + z\|v_{F,t,z} - u_{F,t,z}\|^2 \le \delta_{F,t,z}, \tag{26}$$

*and $\|v_{F,t,z} - u\|^2 \le [Q(u) + \delta_{F,t,z}]/z$. No balancing condition between $F, t, z$ is required.*

*Proof.* The closed form $e_{F,t} + z\|\cdot\|^2$ is coercive on its form Hilbert space. Riesz representation gives its unique minimizer. Using $u$ as competitor gives

$$e_{F,t}(u_{F,t,z}) + z\|u_{F,t,z} - u\|^2 \le e_{F,t}(u) \le Q(u). \tag{27}$$

This proves the state bound and uniform boundedness of the response vectors in $\mathcal{Y}$.

We prove their weak convergence without selecting a subsequence. For $h \in V$, $\langle A^{1/2}u_{F,t,z}, h\rangle = \langle u_{F,t,z}, A^{1/2}h\rangle \to \langle A^{1/2}u, h\rangle$. For each fixed $j$, the index eventually contains $j$, and

$$\langle B_t P_j u_{F,t,z}, h\rangle = \langle P_j u_{F,t,z}, (I + tA)^{-1/2}A^{1/2}h\rangle \longrightarrow \langle A^{1/2}P_j u, h\rangle.$$

Here $(I + tA)^{-1/2}A^{1/2}h \to A^{1/2}h$: the spectral multiplier converges uniformly to one on each bounded interval and is bounded by one everywhere. A fixed spectral cutoff followed by $t \downarrow 0$ proves the assertion, also after pullback to any proper filter. State convergence and boundedness of these test vectors justify the displayed limit. Density of $V$ in $H$, then density of finite-support vectors in the Hilbert direct sum, prove weak convergence in $\mathcal{Y}$. The same argument is valid for uncountable $J$.

Weak lower semicontinuity and (27) now give

$$\|Du\|^2 \le \liminf \|D_{F,t}u_{F,t,z}\|^2 \le \limsup \|D_{F,t}u_{F,t,z}\|^2 \le Q(u) = \|Du\|^2.$$

In a Hilbert space weak convergence together with convergence of norms implies strong convergence. This proves (25). All liminf and limsup statements here are along the specified tail filter.

The Euler equation for the exact minimizer and the quadratic expansion of (23) give the exact identity

$$J_{F,t,z}(v) - J_{F,t,z}(u_{F,t,z}) = \|D_{F,t}(v - u_{F,t,z})\|^2 + z\|v - u_{F,t,z}\|^2.$$

This proves (26). The competitor estimate with the extra $\delta$ proves the other bound. The triangle inequality and (26) give the strong response conclusion for approximate minimizers. There is no assumption that an approximate minimizer already belongs to $V_*$. □

The construction recovers more than a scalar energy along an arbitrary sequence: the output explicitly includes every weighted conditional response. It is, however, a form-graph construction in standard analysis, not an alternative to Mosco convergence. Its target is the conditional energy $Q$, which can have a strictly smaller domain than $q$. It does not claim conditional recovery for a target outside $V_*$.

**Corollary 7.3** (Response cross terms and covariance)**.** *Apply Theorem 7.2 to finitely many targets $u^1, \dots, u^r \in V_*$ on the same directed product. If $L, M$ are bounded operators from $\mathcal{Y}$ to a Hilbert space $Z$, then*

$$\langle LD_{F,t}u^a_{F,t,z}, MD_{F,t}u^b_{F,t,z}\rangle_Z \longrightarrow \langle LDu^a, MDu^b\rangle_Z.$$

*Thus every fixed finite conditional-response Gram matrix converges, including its off-diagonal entries. Bounded synthesis of infinitely many response channels has the same property. If $Z = L^2(\Omega, \mu)$, applying the bounded centering operator gives convergence of the corresponding statistical covariances.*

*Proof.* For strongly converging vectors $x_i \to x$, $y_i \to y$, the estimate $|\langle x_i, y_i\rangle - \langle x, y\rangle| \le \|x_i - x\|\|y_i\| + \|x\|\|y_i - y\|$ proves the assertion. A bounded synthesis map is required: an unbounded sum of channels is not justified by convergence in the direct sum. □

**Theorem 7.4** (Resolvents, exact refinement defect, and domain loss)**.** *Fix $z > 0$ and $f \in H$, and put $x_i = (A_i + z)^{-1}f$ for $i = (F, t)$. If $i \preceq k$, then*

$$e_i(x_i - x_k) + z\|x_i - x_k\|^2 = \langle f, x_i - x_k\rangle - (e_k - e_i)(x_k). \tag{28}$$

*Along the full directed set,*

$$x_i \longrightarrow (A_* + z)^{-1}P_K f =: x, \qquad D_i x_i \longrightarrow Dx \quad \text{strongly.}$$

*Here $P_K$ is the orthogonal projection onto $K = \overline{V_*}$. In particular these are generalized strong resolvents on $H$ when $K \neq H$; a densely defined limiting operator on $H$ is not asserted. The response bounds*

$$\|x_i\| \le \|f\|/z, \qquad \|D_i x_i\| \le \|f\|/(2\sqrt{z})$$

*are uniform in $i$. Both convergence statements persist for forcing $f_i \to f$ strongly and are uniform for $f$ in a compact subset of $H$.*

*Proof.* Let $e_i(\cdot, \cdot)$ denote the polar form. The two Euler equations give $e_i(x_i, x_k) + z\langle x_i, x_k\rangle = \langle f, x_k\rangle$ and $e_k(x_k) + z\|x_k\|^2 = \langle f, x_k\rangle$. Expansion proves (28). In particular $r_i := \langle f, x_i\rangle$ is decreasing and nonnegative, with infimum $r$. For any $i, k$ beyond $i_0$, choose $l \succeq i, k$. Then

$$\|x_i - x_k\| \le \|x_i - x_l\| + \|x_k - x_l\| \le 2\sqrt{(r_{i_0} - r)/z}.$$

Completeness of $H$ gives a strong limit $x$, and $r = \langle f, x\rangle$. For fixed $i$, lower semicontinuity and the cofinal tail $k \succeq i$ give $e_i(x) \le r - z\|x\|^2$. Taking the supremum shows $x \in V_*$ and $Q(x) \le r - z\|x\|^2$. The minimizing inequality at each $i$, tested with $v \in V_*$, gives after passage to the limit

$$-r \le Q(v) + z\|v\|^2 - 2\langle f, v\rangle.$$

For $v = x$ the reverse inequality follows from the preceding bound, so $x$ is the unique minimizer of this limiting functional and $Q(x) = r - z\|x\|^2$. This identifies the generalized resolvent and the convergence of response norms. Coordinatewise weak convergence, proved exactly as in Theorem 7.2, therefore implies $D_i x_i \to Dx$.

Finally the Euler equation gives $e_i(x_i) \le \|f\|\|x_i\| - z\|x_i\|^2 \le \|f\|^2/(4z)$, and also $\|x_i\| \le \|f\|/z$. The maps $f \mapsto x_i$ and $f \mapsto D_i x_i$ are linear with these uniform norm bounds. A triangle inequality proves compatibility with $f_i \to f$. Finite nets in a compact set prove uniform convergence on that set. No operator-norm convergence on the whole unit ball is claimed. □

## 7.1 Exactly when ordinary form recovery suffices

**Theorem 7.5** (Universal conditional compatibility criterion)**.** *The following statements are equivalent:*

1. $V_* = V$;
2. *there is a finite* $C$ *such that* $\sum_j w_j q(P_j u) \le C(\|u\|^2 + q(u))$ *for every* $u \in V$;
3. *every sequence* $u_n \in V$ *converging to* $u \in V$ *in the* $q$ *graph norm has defined conditional response vectors* $(\sqrt{w_j} A^{1/2} P_j u_n)_j$ *and* $(\sqrt{w_j} A^{1/2} P_j u)_j$*, and these converge strongly.*

*When these statements hold, the same preservation is valid for nets along every proper filter. For a fixed closed nonnegative form,* $u_i \to u$ *in* $H$ *and* $q(u_i) \to q(u) < \infty$ *are equivalent to graph-norm convergence. Thus the criterion states exactly when every ordinary energetic recovery net automatically preserves all the weighted conditional responses.*

*Proof.* On the Hilbert space $(V, \|\cdot\|_q)$, the conditional response operator is closed by the proof of Theorem 7.1. If its domain is all of $V$, the closed graph theorem makes it bounded, which is statement (2). That bound gives (3) and the assertion for arbitrary nets. Constant sequences in (3) imply (1). For the last assertion, state convergence and eventual boundedness of $A^{1/2}u_i$ give weak derivative convergence by testing against $V$; convergence of $q(u_i)$ gives convergence of its norm. Hilbert geometry then gives strong derivative convergence. The converse is immediate. □

A useful verifiable sufficient condition for this equivalence is

$$P_j V \subset V, \quad \|A^{1/2}P_j u - P_j A^{1/2} u\| \le c_j \|u\|, \qquad \sum_j w_j < \infty, \quad \sum_j w_j c_j^2 < \infty.$$

Indeed summing $q(P_j u) \le 2q(u) + 2c_j^2\|u\|^2$ proves (2). Commutation with the spectral resolution of $A$ permits $c_j = 0$. The criterion itself is stronger than this sufficient commutator test: no bounded state-space commutator is necessary.

**Corollary 7.6** (A quantitative response rate)**.** *Assume the equivalent conditions of Theorem 7.5, and choose* $C \ge 1$ *such that*

$$Q(v) \le C(\|v\|^2 + q(v)) \qquad (v \in V).$$

*Then $K = H$. For $u \in \operatorname{Dom} A_*$, $z \geq 1$, and $i = (F, t)$, put $r_i(u) = Q(u) - e_i(u) \geq 0$. The anchored minimizers of Theorem 7.2 obey*

$$\|D_i u_{i,z} - Du\| \leq \frac{\|A_* u\|}{\sqrt{z}} + (\sqrt{C} + 1)\sqrt{r_i(u)}. \tag{29}$$

*An approximate objective gap at most $\delta_{i,z}$ adds $\sqrt{\delta_{i,z}}$ to the right-hand side. The two error terms have distinct verifiable meanings: source regularity of the target for $A_*$ and its unresolved weighted conditional energy.*

*Proof.* Let $x = u_{i,z} - u$ and $h^2 = e_i(x) + z\|x\|^2$. Here $V_* = V$, so every term below is on the common domain. Write $r_i(\cdot, \cdot) = Q(\cdot, \cdot) - e_i(\cdot, \cdot)$; this is a nonnegative bilinear form. The Euler equation gives

$$h^2 = -e_i(u, x) = -\langle A_* u, x\rangle + r_i(u, x).$$

Cauchy–Schwarz for the positive form $r_i$ and $Q(x) \leq C(\|x\|^2 + q(x)) \leq Ch^2$, using $z \geq 1$, imply

$$h^2 \leq \|A_* u\| h/\sqrt{z} + \sqrt{r_i(u)}\sqrt{r_i(x)} \leq [\|A_* u\|/\sqrt{z} + \sqrt{C r_i(u)}]h.$$

Dividing when $h > 0$, and treating $h = 0$ directly, bounds $h$. For $v \in V$, spectral calculus and $(1 - s)^2 \leq 1 - s^2$ for $0 \leq s \leq 1$ yield

$$\|B_t v - A^{1/2} v\|^2 \leq q(v) - \|B_t v\|^2.$$

Apply this with $v = P_j u$ for $j \in F$, retain the full missing energies for $j \notin F$, and sum. The base derivative coordinate agrees exactly, so $\|D_i u - Du\|^2 \leq Q(u) - e_i(u) = r_i(u)$. Finally $\|D_i u_{i,z} - Du\| \leq \|D_i x\| + \|D_i u - Du\| \leq h + \sqrt{r_i(u)}$ proves (29). Equation (26) proves the approximate-solver assertion. □

### 7.2 An actual conditional domain obstruction

Let $H = L^2(0, 1)$, let $A$ be the Neumann Laplacian, and hence $V = H^1(0, 1)$ and $q(u) = \int_0^1 |u'|^2$. Let $P$ be conditional expectation onto the partition $\{(0, 1/2), (1/2, 1)\}$ with Lebesgue probability. Then

$$Pu = 2\left(\int_0^{1/2} u\right) 1_{(0,1/2)} + 2\left(\int_{1/2}^1 u\right) 1_{(1/2,1)}.$$

A two-step function is in $H^1(0, 1)$ exactly when its two values agree. For one observation of weight one this gives

$$V_* = \left\{u \in H^1(0, 1) : \int_0^{1/2} u = \int_{1/2}^1 u\right\}.$$

It is dense in the proper closed hyperplane $K = \{u \in L^2 : \langle u, 1_{(0,1/2)} - 1_{(1/2,1)}\rangle = 0\}$. To check density, approximate a point of $K$ by smooth functions and subtract its vanishing scalar moment times a fixed smooth function normalized to have moment one. On $V_*$, $Pu$ is constant and $Q(u) = q(u)$. Every regularized finite form still has full domain $H^1$. Thus the correction in Theorem 7.4 is essential: the limiting variational problem imposes a genuine linear constraint. Neither an $L^2$ contraction nor an increasing sigma-algebra alone ensures preservation of an unbounded derivative energy.

### 7.3 Directed structure and infinitesimal records

The product index of Theorem 7.2 is a standard directed set. Every map from another filtered index that tends to its tail filter gives the same recovery conclusion. Mere cofinality of a nonmonotone map does not suffice; the eventual-tail condition is the one used. If $J$ is countable, an exhaustion $F_n$, together with $t_n \downarrow 0$ and $z_n \uparrow \infty$, gives a Peano-indexed realization. If $J$ is uncountable, no sequence of finite sets exhausts all its labels, so the full directed construction can be necessary for that specified family of observations. Each individual vector in the direct sum has countable support; this does not make the entire observation system countably cofinal.

The scale germs $[t]$ and $[z^{-1}]$ are ordinary asymptotic classes along this filter. In the exact germ ring they are nonzero whenever the positive representatives are eventually defined. They do not assert the existence of infinitesimal real numbers. The estimate (24) is an exact inequality of representatives, which can subsequently be recorded at any chosen asymptotic resolution. Completeness enters through the Hilbert Cauchy argument and the real infimum in Theorem 7.4; it supplies no extra compactness, conditional estimates, or nonstandard transfer.

### 7.4 An intrinsic necessary-and-sufficient recovery test

The preceding construction also characterizes admissible targets without assuming that their unregularized conditional responses exist. Let $\mathcal{I}$ denote the directed observation–regularization index.

**Theorem 7.7** (Intrinsic conditional recoverability)**.** *For any $u \in H$, define $u_{F,t,z}$ as the unique minimizer of $e_{F,t}(v) + z\|v - u\|^2$ on $V$. This minimizer exists for every $u \in H$ by Riesz representation in the shifted form Hilbert space, without assuming $u \in V_*$. Then the following are equivalent:*

1. *$u \in V_*$.*
2. *There is a proper filtered index $\mathcal{L}$, a map $i : \mathcal{L} \to \mathcal{I}$ tending to the tail filter of $\mathcal{I}$, and $v_\ell \in V$ such that $v_\ell \to u$ in $H$ and $e_{i(\ell)}(v_\ell)$ is eventually bounded.*
3. *The canonical anchored net just defined, along the full independent product $F \uparrow J$, $t \downarrow 0$, $z \uparrow \infty$, converges to $u$ and its identified responses converge strongly in $\mathcal{Y}$.*

*In (2), if the responses have a strong limit $y$, then $y = Du$. Consequently every full-response recovery limit is uniquely determined by the state, even though finite-stage conditional domains may differ from the limiting domain.*

*Proof.* Theorem 7.2 proves (1)⇒(3), and strong convergence of responses gives (3)⇒(2). For (2)⇒(1), choose an eventual bound $M$ and fix $i_0 \in \mathcal{I}$. Eventually $i(\ell) \geq i_0$, so $e_{i_0}(v_\ell) \leq M$. Lower semicontinuity in $H$ gives $e_{i_0}(u) \leq M$. This argument for all $i_0$ yields $Q(u) = \sup_{i_0} e_{i_0}(u) \leq M$, proving $u \in V_*$. This use of lower semicontinuity is valid for proper filters and does not require a sequential extraction.

For uniqueness of the response, convergence of the first component and closedness of $A^{1/2}$ identify it as $A^{1/2}u$. For each fixed $j$, the index eventually includes $j$. Testing its component divided by $\sqrt{w_j} > 0$ against $h \in V$ gives

$$\langle B_{t(\ell)} P_j v_\ell, h\rangle = \langle P_j v_\ell, (I + t(\ell)A)^{-1/2} A^{1/2} h\rangle \longrightarrow \langle A^{1/2} P_j u, h\rangle.$$

The spectral test vector converges in $H$, as already proved in Theorem 7.2. Density of $V$ identifies every component of $y$ with that of $Du$. Equality of all coordinates in the Hilbert direct sum gives $y = Du$. □

The eventual bound in (2) is a finite-stage quantity. The theorem therefore permits an admissibility test through approximating problems without applying an unbounded derivative to a conditional observation at each stage. It is a consequence of the explicit monotone realization and its recovery theorem, not a general replacement for compactness or equicoercivity.

## 8 Independent regularizations preserve the obstruction

The counterexample can be reached through bounded coercive energies that retain the same uniform binary bound. In fact the energies converge in norm resolvent sense, and their identified square-root responses recover every target in the limiting form domain. These stronger approximation properties do not supply a coercivity anchor.

**Theorem 8.1** (Binary-cost preserving two-parameter recovery)**.** *Let $L$ be an injective nonnegative selfadjoint operator on a real Hilbert space $H_0$. Let $\mathcal{V}$ be a family of unit vectors in $\operatorname{Dom} L^{1/2} \cap \operatorname{Dom} L^{-1/2}$. For $s, t > 0$ put*

$$\phi_{s,t}(\lambda) = \frac{\lambda + s}{1 + t(\lambda + s)}, \qquad L_{s,t} = \phi_{s,t}(L), \qquad D_{s,t} = L_{s,t}^{1/2}.$$

*Then*

1. *$L_{s,t}$ is bounded and strictly coercive, with $[s/(1+ts)]I \le L_{s,t} \le t^{-1}I$.*
2. *For every $v \in \mathcal{V}$, the exact rank-one energy costs satisfy $K_{L_{s,t}}(v) \le K_L(v)$.*
3. *For every $z > 0$,*
$$\|(L_{s,t} + z)^{-1} - (L + z)^{-1}\| \le t + s/z^2. \tag{30}$$
4. *Along the independent product $s \downarrow 0$, $t \downarrow 0$, $D_{s,t}u \to L^{1/2}u$ strongly for every $u \in \operatorname{Dom} L^{1/2}$. Consequently all fixed finite response Gram matrices, including cross terms, converge.*

*Applied on the mean-zero space in Theorem 5.2 and extended by zero on constants, these forms have the same uniform $(1+\varepsilon)$ binary bound at every stage, but their limit is unbounded and has no positive mean-zero spectral lower bound.*

*Proof.* The first assertion follows directly from the increasing scalar function $\phi_{s,t}$. If $\nu$ is the spectral probability measure of a unit vector $v$ in both half-domains, Tonelli's theorem gives

$$K_L(v) - 1 = \frac{1}{2} \int\int_{(0,\infty)^2} \frac{(a-b)^2}{ab}\, d\nu(a) d\nu(b). \tag{31}$$

Indeed expansion of the integrand gives $a/b + b/a - 2$, whose integrals are finite by the two domain hypotheses. Injectivity removes an atom at zero. For the transformed energy, the same spectral measure and the exact identity

$$\frac{(\phi_{s,t}(a) - \phi_{s,t}(b))^2}{\phi_{s,t}(a)\phi_{s,t}(b)} = \frac{(a-b)^2}{(a+s)(b+s)(1+t(a+s))(1+t(b+s))} \le \frac{(a-b)^2}{ab} \tag{32}$$

prove the cost contraction. The new half-domains are the entire space, since $L_{s,t}$ is bounded above and below.

To prove the resolvent rate, first shift by $s$. For $\lambda \ge 0$,

$$\left| \frac{1}{\lambda + s + z} - \frac{1}{\lambda + z} \right| \le s/z^2.$$

For any $x \geq 0$, direct subtraction gives

$$\frac{1}{x/(1+tx)+z} - \frac{1}{x+z} = \frac{tx^2}{(x+z)(x+z+tzx)} \leq t.$$

Take $x = \lambda + s$ and use the spectral supremum bound and the triangle inequality to obtain (30).

Finally, for fixed $u$ in the limiting form domain the squared response error is

$$\int_{[0,\infty)} |\sqrt{\phi_{s,t}(\lambda)} - \sqrt{\lambda}|^2 \, d\mu_u(\lambda).$$

On every finite spectral interval the integrand tends uniformly to zero as $s, t \downarrow 0$ independently. Eventually $s \leq 1$, and $\phi_{s,t}(\lambda) \leq \lambda + 1$, so the integrand is bounded by $4\lambda + 2$. Choose a spectral cutoff with small integral of this bound outside it, then use uniform convergence inside. This establishes the limit for the product filter without invoking an unrestricted dominated-convergence theorem for nets. Continuity of the Hilbert inner product proves the Gram assertion. □

This is a FINER construction in ordinary standard analysis: its directed parameters are $s, t > 0$ ordered toward zero, together with finite sets of response tests if desired. The binary-cost and norm-resolvent estimates are uniform in those tests and do not require a relation between the two scales. Strong response recovery uses a spectral cutoff depending on the fixed target; it is not asserted uniformly on arbitrary bounded sets of form-domain vectors. The associated vanishing germs record the estimates but do not prove them. A countable cofinal grid exists for these two real scales; ordinary diagonal sequences can represent this particular limit. Thus filtered notation is useful for the stated uniformity, but is not mathematically necessary for the existence of this example. The obstruction is analytic: the positive lower constants $s/(1+ts)$ tend to zero and the upper constants $1/t$ diverge, while the binary tests remain uniformly almost contractive.

## 9 Identified responses beyond scalar form convergence

Let $H, Y$ be real Hilbert spaces, and let $K_i, K$ be closed subspaces of $H$. The index $i$ ranges over a set equipped with a fixed proper filter $\mathcal{F}$; all limits in this section are taken along that filter. Consider closed linear operators $D_i : V_i \subset K_i \to Y$ and $D : V \subset K \to Y$ with domains dense in their respective state spaces. Their forms, extended by $+\infty$ outside their domains, are $q_i(u) = \|D_i u\|^2$ and $q(u) = \|Du\|^2$. Write $A_i = D_i^* D_i$ on $K_i$ and $A_* = D^* D$ on $K$. Adjoints here are taken from $Y$ into $K_i$ or $K$, and their values are embedded in $H$. Suppose

$$R_i(z) := (A_i + z)^{-1} P_{K_i} \longrightarrow R(z) := (A_* + z)^{-1} P_K \quad \text{strongly}, \qquad z > 0. \tag{33}$$

This is the generalized strong-resolvent condition supplied by scalar Mosco convergence. One positive shift suffices, by the resolvent identity and uniform bounds. Functional calculus then gives convergence of $\varphi(A_i)P_{K_i}$ for every $\varphi \in C_0([0,\infty))$. Indeed, transform $\varphi$ into a continuous function of $(\lambda + z)^{-1}$ which vanishes at zero, approximate it uniformly by polynomials with zero constant term, and use strong convergence and uniform boundedness of the resolvents. This accounts for the possibly nondense limiting space.

Let $D_i = U_i A_i^{1/2}$ and $D = U A_*^{1/2}$ be the polar representations, extending $U_i, U$ by zero outside $K_i, K$. Set

$$M = K \ominus \ker A_*, \qquad N = \overline{\operatorname{ran} D}.$$

The maps $U_i : H \to Y$ are contractions, and $U$ is an isometry from $M$ onto $N$.

**Theorem 9.1** (Active response characterization)**.** *Under* (33)*, the following are equivalent.*

1. *For every $u \in V$ and every family $u_i \in V_i$ with $u_i \to u$ in $H$ and $q_i(u_i) \to q(u)$, one has $D_i u_i \to Du$ strongly in $Y$.*
2. *For one $z > 0$ and every $f \in H$, $D_i R_i(z) f \rightharpoonup DR(z)f$ in $Y$.*
3. *For one $z > 0$ and every $f \in H$, the convergence in (2) is strong.*
4. *$U_i x \to Ux$ for every $x \in M$.*
5. *There is a dense set $C \subset N$ with $C \subset \operatorname{Dom} D^*$ such that each $y \in C$ has $y_i \in \operatorname{Dom} D_i^*$ satisfying*
$$y_i \to y \quad \text{in } Y, \qquad D_i^* y_i \to D^* y \quad \text{in } H.$$

*When they hold, (2)–(3) hold for every positive shift. In (4), weak convergence is equivalent to strong convergence. Dense forcing and test sets suffice in (2)–(3); in particular countable dense tests suffice when the spaces are separable. The adjoint test set in (5) covers only the active response space $N$.*

*Proof.* Functional calculus yields

$$G_i(z) := A_i^{1/2} R_i(z) \longrightarrow G(z) := A_*^{1/2} R(z), \qquad \|D_i R_i(z)\| \le \frac{1}{2\sqrt{z}}. \tag{34}$$

The multiplier $\sqrt{\lambda}/(\lambda + z)$ lies in $C_0$ and has supremum $1/(2\sqrt{z})$. Its nonzero spectral support is $(0, \infty)$, so $\overline{\operatorname{ran} G(z)} = M$. The Euler identity gives

$$\|D_i R_i(z) f\|^2 = \langle f, R_i(z) f \rangle - z \|R_i(z) f\|^2 \longrightarrow \|DR(z) f\|^2. \tag{35}$$

This proves (2) implies (3) by Hilbert geometry; the converse is immediate. Because

$$\|U_i G_i(z) f - U_i G(z) f\| \le \|G_i(z) f - G(z) f\| \to 0,$$

(3) is equivalent to convergence of $U_i$ to $U$ on $\operatorname{ran} G(z)$. Uniform contractivity extends it to $M$: approximate $x \in M$ by $G(z)f$ and use

$$\|U_i x - Ux\| \le 2\|x - G(z) f\| + \|U_i G(z) f - UG(z) f\|.$$

This proves (3) iff (4), independently of $z$. Weak convergence in (4) also implies strong convergence, since $\|U_i x\| \le \|x\| = \|Ux\|$ on $M$.

Assume (4), fix $z > 0$, and take $C = \{UR(z)f : f \in M\}$. This is dense in $N$ because $R(z)|_M$ has dense range in $M$ and $U|_M$ is an isometry. For $y = UR(z)f$, set $y_i = U_i R_i(z) f$. Strong resolvent convergence and (4) imply $y_i \to y$. Moreover

$$D_i^* y_i = A_i^{1/2} R_i(z) f \to A_*^{1/2} R(z) f = D^* y.$$

For the domain assertions, write $U_i^* U_i = P_{K_i} - P_{\ker A_i}$. This spectral projection preserves $\operatorname{Dom} A_i$, which contains $R_i(z) f$. Consequently $U_i^* y_i \in \operatorname{Dom} A_i^{1/2}$, and the adjoint formula $D_i^* = A_i^{1/2} U_i^*$ applies to $y_i$. The same argument applies to $y$ and $D^*$. Thus (4) implies (5).

Suppose (5) and take a recovery family in (1). Its response norms are eventually bounded and converge to $\|Du\|$. For the actual adjoint tests,

$$\langle D_i u_i, y_i \rangle = \langle u_i, D_i^* y_i \rangle \longrightarrow \langle u, D^* y \rangle = \langle Du, y \rangle.$$

Replacing $y_i$ by $y$ is legitimate by their eventual boundedness. Density of $C$ in $N$ permits approximation of the single vector $Du$, giving $\langle D_i u_i, Du\rangle \to \|Du\|^2$. Expanding $\|D_i u_i - Du\|^2$ proves (1). Finally $u_i = R_i(z)f$ is an energetic recovery net for $R(z)f$ by (33) and (35), so (1) implies (3). All arguments use filter events and fixed approximations; no subsequence or interchange of a general net and an integral is required. The uniform bound in (34) proves the dense-test assertions. □

Condition (2) can equivalently be written as weak convergence of $[D_i R_i(z)]^* y$ for every $y \in Y$. This is the bounded extension of $R_i(z)D_i^*$; the latter expression cannot be applied to an arbitrary $y$ without specifying that extension. The theorem does not require strong convergence of these adjoints on all of $Y$.

For canonical positive square roots, the polar factors are $P_{K_i} - Q_i$, where $Q_i = P_{\ker A_i}$. Strong resolvent convergence gives $P_{K_i} x \to x$ for $x \in K$, by approximation with limiting resolvent ranges. For $x \in M$,

$$\|Q_i x\|^2 \le \langle x, (I + sA_i)^{-1} P_{K_i} x\rangle;$$

first pass to the limit for fixed $s$, then let $s \to \infty$. This gives $Q_i x \to 0$, hence condition (4). Thus canonical square-root recovery is already a consequence of scalar form convergence. A specified response has the additional orientation $U_i$. Strong response convergence preserves every boundedly synthesized Gram matrix, including all cross terms; centering in $L^2$ gives covariance convergence.

**Necessity examples.** On $H = Y = \mathbb{R}$, take $D_n = (-1)^n I$, $D = I$. All scalar forms and all simultaneous response Gram matrices agree, but the responses of the target 1 do not converge. Therefore scalar energies and even complete unanchored Gram data do not identify orientation. Taking $H = \mathbb{R}$, $Y = \ell^2$, $D_n u = u e_n$ and $Du = u e_1$ additionally shows that correct response norms can coexist with a wrong weak limit.

Full graph convergence is stronger than the theorem. On $H = \mathbb{R} \oplus \ell^2$ and $Y = \mathbb{R}^2$, let

$$D_n(a, x) = (a, n x_n), \qquad D(a, x) = (a, 0).$$

The scalar resolvents converge strongly: the only nonconstant resolvent discrepancy in the $\ell^2$ part is bounded by a constant times $|x_n|$ for each fixed $x$. The active polar maps agree on $M = \mathbb{R} \oplus \{0\}$, so the theorem applies. However $(0, e_n/n) \to 0$ while its response is $(0, 1)$. This is a bounded-energy net, not an energetic recovery of zero. Moreover any $y_n \to (0, 1)$ satisfies $\|D_n^* y_n\| \ge n|(y_n)_2| \to \infty$. Thus adjoint recovery tests dense in all of $Y$ would be an unnecessary hypothesis. The active set $\mathbb{R} \times \{0\}$ works.

## 10 Dynamic recovery and cumulative proximal time

The conditional completion produces a closed form $Q$ on $K = \overline{\operatorname{Dom} Q}$ and its actual response operator $D : \operatorname{Dom} Q \subset K \to \mathcal{Y}$, with $Q(u) = \|Du\|^2$. Let $A \ge 0$ be the self-adjoint operator of $Q$ on $K$. Thus $Q(u) = \|A^{1/2} u\|^2$. The completed dynamics first considered below take place on $K$; the simultaneous approximation theorem also treats finite-stage dynamics on $H$. In particular no density of the original conditional domain in a larger state space is assumed.

For a finite list $\pi = (\tau_1, \dots, \tau_m)$, $m \ge 1$, of positive numbers, set

$$R_k = (I + \tau_k A)^{-1}, \qquad P_\pi = R_m \cdots R_1, \qquad S_\pi = \sum_{k=1}^{m} \tau_k, \qquad h_\pi = \max_{1 \le k \le m} \tau_k.$$

Each resolvent step is the unique minimizer of $Q(v)+\tau_k^{-1}\|v-x\|^2$ with input $x$. The number $h_\pi$ is the upper time mesh; $S_\pi$ is the cumulative proximal time. They are independent pieces of convergence data. The finite lists may themselves form a net along any proper filter; no ordering of their lengths or individual entries is assumed.

For clarity, *fidelity to a target* in this section means small state distance $\|P_\pi u-u\|$, and, when $u\in\operatorname{Dom}Q$, small response distance $\|D(P_\pi u-u)\|$. It does not mean a quantum transition probability or an unverified probabilistic metric. Backward Euler products and their spectral calculus are classical; unequal-step semigroup approximations are treated in Bernád and Frigyik [5]. The purpose here is to characterize exactly which cumulative time produces fidelity, a finite-time evolution, or complete damping, and to carry the previously specified conditional responses through the same construction.

## 10.1 Exact defects and three distinct limits

Let $E_A$ be the spectral resolution of $A$, let $\mu_u(B)=\|E_A(B)u\|^2$, and put

$$p_\pi(\lambda)=\prod_{k=1}^{m}(1+\tau_k\lambda)^{-1}.$$

The following elementary bounds hold for $\lambda\ge 0$:

$$e^{-\lambda S_\pi}\le p_\pi(\lambda)\le(1+\lambda S_\pi)^{-1}, \tag{36}$$

$$0\le\lambda S_\pi+\log p_\pi(\lambda)\le\frac{\lambda^2}{2}h_\pi S_\pi. \tag{37}$$

Indeed $\log(1+x)\le x$, the expansion of the positive product gives $\prod_k(1+\tau_k\lambda)\ge 1+\lambda S_\pi$, and $0\le x-\log(1+x)\le x^2/2$ for $x\ge 0$.

**Theorem 10.1** (Cumulative-time characterization)**.** *Let $u\in K\setminus\ker A$ be fixed. Along any proper filter of finite positive step lists:*

1. *$P_\pi u\to u$ strongly if and only if $S_\pi\to 0$.*
2. *$P_\pi u\to E_A(\{0\})u$ strongly if and only if $S_\pi\to\infty$.*
3. *Fix $T\in(0,\infty)$ and assume $S_\pi\to T$. Then $P_\pi u\to e^{-TA}u$ strongly if and only if $h_\pi\to 0$.*

*If $u\in\operatorname{Dom}Q$, every stated sufficient condition also implies convergence of the actual response $DP_\pi u$ to the response of the stated limit. The necessity assertions remain true when state and response convergence are required together. If $u\in\ker A$, all step products fix $u$, so the three necessity assertions cannot be made for that target.*

*For $T\ge 0$ the state and response defects have the exact formulas*

$$\|(P_\pi-e^{-TA})u\|^2=\int_{[0,\infty)}|p_\pi(\lambda)-e^{-T\lambda}|^2\,d\mu_u(\lambda), \tag{38}$$

$$\|D(P_\pi-e^{-TA})u\|^2=\int_{[0,\infty)}\lambda|p_\pi(\lambda)-e^{-T\lambda}|^2\,d\mu_u(\lambda), \tag{39}$$

*where the second formula is asserted for $u\in\operatorname{Dom}Q$. In particular*

$$\|(P_\pi-I)u\|^2\le S_\pi Q(u),\qquad \|D(P_\pi-I)u\|^2\le S_\pi\|Au\|^2, \tag{40}$$

*with domains $u\in\operatorname{Dom}Q$ and $u\in\operatorname{Dom}A$, respectively.*

*Proof.* The functional calculus gives $P_\pi = p_\pi(A)$ and (38). Each resolvent contracts the form norm and preserves its domain. Applying $\|Dv\|^2 = \|A^{1/2}v\|^2$ proves (39); this argument retains the orientation of $D$, not only a separately chosen square root of an energy. By (36),

$$0 \le 1 - p_\pi(\lambda) \le 1 - e^{-\lambda S_\pi}, \qquad (1 - p_\pi(\lambda))^2 \le \lambda S_\pi.$$

Integration proves (40).

For arbitrary nets we justify sufficiency using spectral cutoffs, not an unrestricted dominated-convergence theorem for nets. For $S_\pi \to 0$, the multipliers converge uniformly to one on every bounded spectral interval, by the preceding inequality. Choose a finite interval first so that the remaining $\mu_u$ mass is small, then use this uniform convergence. The weighted measure $\lambda\, d\mu_u$ gives the same argument for the response when it has finite total mass. For $S_\pi \to \infty$, the upper bound in (36) tends uniformly to zero on every interval $[\alpha, R]$ with $0 < \alpha < R < \infty$. The positive spectral mass in $(0, \alpha)$ and in $(R, \infty)$ can first be made small; the atom at zero is fixed exactly. The same argument applies to the weighted measure for the response.

For the finite-time assertion define, for $R > 0$,

$$\eta_{\pi,R} = R|S_\pi - T| + \tfrac{1}{2}R^2 h_\pi S_\pi.$$

The function $s \mapsto e^{-s}$ is 1-Lipschitz on $[0, \infty)$, so (37) gives $|p_\pi(\lambda) - e^{-T\lambda}| \le \eta_{\pi,R}$ for $0 \le \lambda \le R$. Since both multipliers lie in $[0, 1]$,

$$\|(P_\pi - e^{-TA})u\|^2 \le \eta_{\pi,R}^2 \|u\|^2 + \mu_u((R, \infty)), \tag{41}$$

$$\|D(P_\pi - e^{-TA})u\|^2 \le \eta_{\pi,R}^2 Q(u) + \int_{(R,\infty)} \lambda\, d\mu_u. \tag{42}$$

Choosing $R$ first and then an event of the filter proves sufficiency.

For necessity, $u \notin \ker A$ implies that $\mu_u([\alpha, R]) > 0$ for some $0 < \alpha < R < \infty$. If $S_\pi$ fails to tend to zero, there is an $\varepsilon > 0$ for which $S_\pi \ge \varepsilon$ frequently. On this set, (36) gives $1 - p_\pi(\lambda) \ge \alpha\varepsilon/(1 + \alpha\varepsilon)$ for $\lambda \in [\alpha, R]$. The state defect cannot tend to zero. If $S_\pi$ fails to tend to infinity, there is a finite $M$ such that $S_\pi \le M$ frequently. On the same spectral interval, $p_\pi(\lambda) \ge e^{-RM}$, which prevents convergence to the kernel projection.

Finally suppose $S_\pi \to T > 0$ but $h_\pi$ does not tend to zero. On a frequent set $h_\pi \ge \varepsilon > 0$. The largest step contributes at least $g(\alpha\varepsilon) > 0$ to the logarithmic deficit, where $g(s) = s - \log(1 + s)$; all other contributions are nonnegative. Eventually $S_\pi \le T + 1$, so uniformly on $[\alpha, R]$,

$$p_\pi(\lambda) - e^{-T\lambda} \ge e^{-R(T+1)}(e^{g(\alpha\varepsilon)} - 1) - R|S_\pi - T|.$$

The right side is bounded away from zero on a frequent set after intersecting with a suitable event of the filter. This contradicts strong finite-time convergence. All necessity arguments therefore apply directly to proper filters, without a sequential extraction. □

The exact upper-mesh criterion in part (3) presupposes the independently specified limit $S_\pi \to T$. A product can be tuned to one spectral value by a different cumulative time; no stronger converse is claimed. Parts (1) and (2), in contrast, need no a priori upper bound on the individual steps. The identities prove recovery in the entire conditional-response Hilbert space of the preceding section, and hence preservation of its boundedly synthesized cross covariances.

## 10.2 Simultaneous conditional refinement and dynamics

The preceding theorem keeps the completed energy fixed. The next result lets the conditional-observation approximation and the time mesh vary independently. It is here that the common response space in Theorem 7.4 is used quantitatively.

**Theorem 10.2** (Joint conditional and temporal recovery)**.** *Let $e_i$, $D_i$, $Q$, $D$, and $K$ be the conditional completion of Theorem 7.1, with $i = (F, t)$ tending to its refinement tail. Write $A_i$ for the densely defined operator of $e_i$ on $H$ and $A_*$ for the operator of $Q$ on $K$. Fix $T > 0$. Let finite nonempty step lists $\pi$ satisfy $S_\pi \to T$ and $h_\pi \to 0$. For every $f \in H$, along the independent joint refinement,*

$$p_\pi(A_i)f \longrightarrow e^{-TA_*}P_K f \qquad \text{strongly in } H, \tag{43}$$

$$D_i p_\pi(A_i)f \longrightarrow De^{-TA_*}P_K f \qquad \text{strongly in } \mathcal{Y}. \tag{44}$$

*No balance between the conditional index $i$ and the time mesh is required. The conclusions also hold for inputs $f_{i,\pi} \to f$ strongly. In particular all boundedly synthesized response Gram matrices converge with their cross terms retained.*

*The purely temporal error is uniform over $i$. More precisely, put*

$$\varepsilon_\pi(T) = \sup_{\lambda \geq 0}(1+\lambda)|p_\pi(\lambda) - e^{-T\lambda}|. \tag{45}$$

*Then $\varepsilon_\pi(T) \to 0$, and*

$$\begin{aligned} \|(p_\pi(A_i) - e^{-TA_i})f\| &\leq \varepsilon_\pi(T)\|f\|, \\ \|D_i(p_\pi(A_i) - e^{-TA_i})f\| &\leq \tfrac{1}{2}\varepsilon_\pi(T)\|f\|. \end{aligned} \tag{46}$$

*Proof.* We first establish the functional calculus required for conditional refinement. Theorem 7.4 gives

$$(A_i + 1)^{-1} \longrightarrow (A_* + 1)^{-1}P_K$$

strongly, with all these operators bounded by one. Powers and polynomials with zero constant term therefore converge strongly. For $\varphi \in C_0([0,\infty))$, define on $[0,1]$

$$g(r) = \begin{cases} \varphi(r^{-1} - 1), & r > 0, \\ 0, & r = 0. \end{cases}$$

This is continuous. Weierstrass approximation, followed by subtracting the approximating polynomial's value at zero, approximates $g$ uniformly by polynomials with zero constant term. The uniform functional-calculus bound and a three-term approximation estimate give

$$\varphi(A_i) \longrightarrow \varphi(A_*)P_K \quad \text{strongly on } H. \tag{47}$$

This proof also accounts for $K^\perp$: the zero constant term makes all the limiting polynomials vanish there. It does not assert the convergence of functions which fail to vanish at spectral infinity.

In particular $e^{-TA_i}f \to e^{-TA_*}P_K f$. The static response theorem additionally gives the strong convergence

$$L_i := D_i(A_i + 1)^{-1} \longrightarrow L_* := D(A_* + 1)^{-1}P_K, \qquad \|L_i\| \leq \tfrac{1}{2}.$$

Use (47) with $\varphi(\lambda) = (1+\lambda)e^{-T\lambda}$, which is in $C_0$ because $T > 0$. Factoring on the form domain yields

$$D_i e^{-TA_i}f = L_i\varphi(A_i)f \longrightarrow L_*\varphi(A_*)P_K f = De^{-TA_*}P_K f.$$

This proves convergence with the actual $D_i$ orientations. A convergence statement only for their Gram operators would not justify this factorization limit.

It remains to prove the uniform temporal estimate. Choose a fixed $h_0 \in (0, T/4)$. Eventually $h_\pi \le h_0$ and $S_\pi \ge T/2$. For each $\lambda \ge 0$ the function $x \mapsto \log(1+\lambda x)/x$ is decreasing for $x > 0$: its derivative has the sign of $\lambda x/(1+\lambda x) - \log(1+\lambda x) \le 0$. Consequently

$$\sum_k \log(1+\lambda\tau_k) \ge \frac{S_\pi}{h_0}\log(1+\lambda h_0), \qquad p_\pi(\lambda) \le (1+h_0\lambda)^{-T/(2h_0)}.$$

The exponent is larger than two. Hence

$$\sup_{\lambda\ge R}(1+\lambda)\big[(1+h_0\lambda)^{-T/(2h_0)} + e^{-T\lambda}\big] \longrightarrow 0 \quad (R\uparrow\infty),$$

providing a tail bound uniform over this eventual set of meshes. On $[0, R]$, (37) gives the bound $(1+R)\eta_{\pi,R} \to 0$ for the weighted difference in (45). Choosing $R$ first and then a filter event proves $\varepsilon_\pi(T) \to 0$.

Let $g_\pi(\lambda) = (1+\lambda)[p_\pi(\lambda) - e^{-T\lambda}]$. It is bounded, and spectral calculus gives

$$D_i(p_\pi(A_i) - e^{-TA_i})f = L_i g_\pi(A_i) f.$$

This identity is legitimate for every $f \in H$: a nonempty resolvent product maps into the form domain, and its weighted multiplier is bounded. Thus the bound on $L_i$ proves the response estimate in (46). The state estimate follows directly from the supremum norm of the unweighted multiplier. Combining these estimates with the fixed-$T$ conditional-refinement limits proves (43) and (44) on the full independent product filter.

Finally $\|p_\pi(A_i)\| \le 1$, and the response operators are uniformly bounded on an eventual set by $\frac{1}{2}[\sup_{\lambda\ge0}(1+\lambda)e^{-T\lambda} + \varepsilon_\pi(T)]$. A triangle inequality therefore permits strongly varying inputs. Convergence of bounded response Gram matrices follows from strong convergence in the common target. □

The positive time in this theorem supplies smoothing uniformly across all conditional refinements. At time zero the asserted weighted multiplier bound is not available. A simultaneous infinite-time limit also needs more than general strong-resolvent convergence. For example, on $\mathbb{R}$, let $A_n = 1/n$ and take $n^2$ steps of size $1/n$. Then $A_n \to 0$, the upper mesh tends to zero, and cumulative time tends to infinity, but the product tends to $e^{-1}$ rather than the kernel projection of the limiting operator, which is the identity. The increasing conditional forms have additional kernel inheritance; Theorem 11.3 below uses it to prove a joint long-time limit for this specific construction. This is a limitation of drawing joint long-time conclusions from strong resolvent convergence alone. It does not contradict Theorem 10.1, which keeps its generator fixed, and the family in this example is not asserted to be an increasing conditional completion.

### 10.3 Accumulated errors and a weighted energetic budget

Suppose the computed states obey

$$y_0 = u, \qquad y_k = R_k y_{k-1} + r_k. \tag{48}$$

For $1 \le k \le m$ set $P_{>k} = R_m \cdots R_{k+1}$, where $P_{>m} = I$, and $S_{>k} = \sum_{l=k+1}^{m} \tau_l$.

**Theorem 10.3** (Exact accumulation and proximal-gap budget)**.** *For $u, r_k \in K$ the state error is exactly*

$$y_m - P_\pi u = \sum_{k=1}^{m} P_{>k} r_k. \tag{49}$$

*If also $u, r_k \in \operatorname{Dom} Q$, then*

$$\|D(y_m - P_\pi u)\|^2 = \sum_k \|DP_{>k} r_k\|^2 + 2 \sum_{k<l} \langle DP_{>k} r_k, DP_{>l} r_l \rangle. \tag{50}$$

*In particular, arbitrary positive weights $a_k$ give*

$$\|D(y_m - P_\pi u)\|^2 \le \left( \sum_k a_k \right) \left( \sum_k \frac{\|DP_{>k} r_k\|^2}{a_k} \right). \tag{51}$$

*This is a bound on the exact sum, not an assertion that the cross terms vanish.*

*Assume that $y_k \in \operatorname{Dom} Q$ achieves the proximal objective with gap at most $\delta_k \ge 0$:*

$$Q(y_k) + \tau_k^{-1} \|y_k - y_{k-1}\|^2 \le \inf_{v \in \operatorname{Dom} Q} \{ Q(v) + \tau_k^{-1} \|v - y_{k-1}\|^2 \} + \delta_k.$$

*Then the actual defects in (48) satisfy*

$$Q(r_k) + \tau_k^{-1} \|r_k\|^2 \le \delta_k. \tag{52}$$

*Consequently*

$$\|y_m - P_\pi u\| \le \sum_k \sqrt{\tau_k \delta_k}, \tag{53}$$

$$\|D(y_m - P_\pi u)\| \le \sum_k b_k \sqrt{\delta_k}, \tag{54}$$

$$b_k = \begin{cases} \min\{1, \sqrt{\tau_k / (4 S_{>k})}\}, & S_{>k} > 0, \\ 1, & S_{>k} = 0. \end{cases}$$

*Thus the respective budgets tending to zero transfer the exact limits of Theorem 10.1 to inexact schemes. The response budget pays the propagation of each defect by the remaining proximal time; no positive spectral gap is required.*

*Proof.* Induction in (48) proves (49). Resolvents preserve the form domain, so applying the linear operator $D$ and expanding its squared norm proves (50). The triangle inequality and scalar weighted Cauchy–Schwarz give (51).

The exact minimizer of step $k$ is $R_k y_{k-1}$. Its variational Euler equation cancels the linear terms in the difference of the two objectives. The difference is exactly $Q(r_k) + \tau_k^{-1} \|r_k\|^2$; hence (52). Each $P_{>k}$ contracts the state norm and the form seminorm. For $S_{>k} > 0$, the upper bound in (36), now applied to the suffix, also gives

$$\|DP_{>k} v\|^2 = \int \lambda p_{>k}(\lambda)^2 \, d\mu_v \le \sup_{\lambda \ge 0} \frac{\lambda}{(1 + \lambda S_{>k})^2} \|v\|^2 = \frac{\|v\|^2}{4 S_{>k}}.$$

This holds for every $v \in K$, with $P_{>k} v$ in the form domain; the displayed bound also proves that domain assertion. For the present $r_k \in \operatorname{Dom} Q$, combine this smoothing bound with contraction and (52) to obtain $\|DP_{>k} r_k\| \le b_k \sqrt{\delta_k}$. Summation proves (54); contraction and $\|r_k\| \le \sqrt{\tau_k \delta_k}$ prove (53). □

The sufficient budgets are not necessary in general: signed defects can cancel. Equation (50) is the exact accounting when such cancellation is justified. No conditional independence of solver errors has been assumed.

### 10.4 Sharp distinctions and scope

On $K = \mathbb{R}$ with $A = 1$ and target $u = 1$, take $m$ equal steps. The choices $\tau_k = m^{-2}$, $m^{-1}$ and $m^{-1/2}$ all have $h_\pi \to 0$, but their outputs tend respectively to 1, $e^{-1}$ and 0. Thus a vanishing upper mesh alone determines none of these three limits. A single step $\tau_1 = 1$ has the correct cumulative time $S_\pi = 1$ but output $1/2 \neq e^{-1}$; cumulative time alone also fails to determine finite-time dynamics.

Small individual optimization gaps need not preserve the final state or response. Again let $A = 1$, now $u = 0$, $\tau_k = 1/m$, and deliberately choose $r_k = 1/m$ at each step. The actual objective gap is $\delta_k = (m+1)/m^2 \to 0$, but

$$y_m = \frac{m+1}{m}\left[1 - \left(\frac{m}{m+1}\right)^m\right] \longrightarrow 1 - e^{-1}.$$

The exact output is zero, and $D$ may be the identity, so both state and response errors persist. The example identifies cumulative error, rather than single-step consistency, as the missing condition.

The proofs above cover unbounded nonnegative self-adjoint $A$, including infinite-dimensional and gapless operators. They do not require a finite-dimensional cutoff. Finite-dimensional discretizations inherit the same statements for their actual forms, but passage between different discretization spaces needs the previously verified identifications and recovery estimates. Each product uses one fixed generator, either $A$ on $K$ or a finite-stage $A_i$ on $H$: its resolvents commute because all are functions of that same generator. Arbitrarily changing forms at successive steps are outside this theorem.

The earlier Dynamic FINER investigations treated a different extension: measurable nonautonomous joint forms, energy-factorized hidden evolution, and response-strong perturbation estimates with the hidden initial state retained. Their common-domain, uniform-coercivity and coefficient-error hypotheses are substantive and are not consequences of the present spectral argument. Conversely, those statements did not establish the three cumulative-time equivalences above. The two results are compatible but neither is substituted for the other's premises. The present article uses only the self-contained autonomous construction just proved; its publication does not depend on the earlier Wilson or nonlinear fluid applications.

## 11 Joint damping and the zero spectral atom

Here the finite-stage operators $A_i$ are densely defined, nonnegative and self-adjoint on $H$, while the generalized limit $A_*$ acts on a possibly proper closed subspace $K$ as in (33). Set

$$Q_i = P_{\ker A_i}, \quad Z = \ker A_*, \quad Q = P_Z.$$

For a finite nonempty positive step list $\pi = (\tau_1, \dots, \tau_m)$, write $S_\pi = \sum_k \tau_k$ and $p_\pi(\lambda) = \prod_k (1 + \tau_k \lambda)^{-1}$. The time-list filter is generated by $S_\pi \geq M$. Independent refinement means its product with the operator-index filter. The scalar bounds are

$$e^{-S_\pi \lambda} \leq p_\pi(\lambda) \leq (1 + S_\pi \lambda)^{-1}, \qquad \lambda p_\pi(\lambda)^2 \leq \frac{1}{4 S_\pi}. \tag{55}$$

They follow from $\log(1+x) \leq x$, expansion of a product of positive factors, and the maximum of $\lambda/(1 + S\lambda)^2$.

**Theorem 11.1** (Independent damping criterion)**.** *Under generalized strong resolvent convergence along the proper operator-index filter $\mathcal{F}$, the following are equivalent:*

1. *$Q_i \to Q$ strongly;*

2. $p_\pi(A_i)f \to Qf$ *for every* $f \in H$ *under independent refinement and* $S_\pi \to \infty$*;*

3. $e^{-tA_i}f \to Qf$ *for every* $f \in H$ *under independent refinement and* $t \to \infty$*;*

4. $\operatorname{dist}(z, \ker A_i) \to 0$ *for every* $z \in Z$*.*

*For* $S_\pi \geq M > 0$*, the quantitative certificate is*

$$\|p_\pi(A_i)f - Qf\| \leq \|Q_i f - Qf\| + \left[\langle f, (I + MA_i)^{-1}f\rangle - \|Q_i f\|^2\right]^{1/2}. \tag{56}$$

*The bracket is nonnegative. No condition on the upper time mesh is needed, and no operator-norm conclusion is asserted.*

*Proof.* Spectral calculus gives

$$\|p_\pi(A_i)f - Q_i f\|^2 = \int_{(0,\infty)} p_\pi(\lambda)^2 \, d\mu_i^f(\lambda) \leq \langle f, (I + MA_i)^{-1}f\rangle - \|Q_i f\|^2.$$

Triangle inequality proves (56). If (1) holds, then for fixed $M$ its bracket tends to

$$\langle P_K f, (I + MA_*)^{-1}P_K f\rangle - \|Qf\|^2,$$

which tends to zero as $M \to \infty$. Choosing $M$ first proves (2) with independent rates. The same argument proves (3), using $e^{-2t\lambda} \leq (1 + M\lambda)^{-1}$ for $t \geq M$.

Conversely, (2) gives, for each fixed $f$ and tolerance, an operator-index event and a time threshold for which all subsequent errors are small. Fix any index in that event and let a one-step time tend to infinity. Since $(I + sA_i)^{-1}f \to Q_i f$, the same error bound holds for $Q_i f - Qf$. This proves (1). The argument for (3) is identical with the fixed-operator heat limit.

For $w \perp Z$,

$$\|Q_i w\|^2 \leq \langle w, (I + MA_i)^{-1}w\rangle.$$

Pass to the limit at fixed $M$ and then let $M \to \infty$ to get $Q_i w \to 0$. Thus only the stationary component requires a further condition, and orthogonal projection gives $\|Q_i z - z\| = \operatorname{dist}(z, \ker A_i)$. This proves (4) equivalent to (1). □

Exact kernel inheritance $Z \subseteq \ker A_i$ is a sufficient special case, giving the original monotone conditional theorem. It is not necessary: rotating rank-one projections on $\mathbb{R}^2$ can have converging kernels without containing the limiting kernel at any finite stage. Small energy on the limiting kernel is weaker: $A_n = n^{-1}I \to 0$ but $Q_n = 0$, $Q = I$, and a one-step time $n$ gives output $f/2$.

**Theorem 11.2** (Coupled spectral survival)**.** *Suppose* $(i, \pi)$ *varies along an arbitrary proper filter whose operator-index projection tends to* $\mathcal{F}$ *and along which* $S_\pi \to \infty$*. Fix* $f \in H$*, and put* $z = Qf$*,* $g = f - z$*. Then* $p_\pi(A_i)g \to 0$*. The following are equivalent:*

$$p_\pi(A_i)f \to Qf, \qquad \|[I - (I + S_\pi A_i)^{-1}]z\| \to 0,$$

*and, for every* $\delta > 0$*,*

$$\|E_i([\delta/S_\pi, \infty))z\| \longrightarrow 0. \tag{57}$$

*Here* $E_i$ *is the spectral resolution of* $A_i$*. The stationary-state defect has the uniform comparison*

$$\|[I - (I + S_\pi A_i)^{-1}]z\| \leq \|[I - p_\pi(A_i)]z\| \leq 2\|[I - (I + S_\pi A_i)^{-1}]z\|. \tag{58}$$

*The factor* 2 *is not asserted optimal.*

*Proof.* For $S_\pi \geq M$, $\|p_\pi(A_i)g\|^2 \leq \langle g, (I + MA_i)^{-1}g\rangle$. The fixed-$M$ limit tends to zero as $M \to \infty$ because $g \perp Z$. Hence the first assertion. The exact decomposition

$$p_\pi(A_i)f - Qf = p_\pi(A_i)g + [p_\pi(A_i) - I]z$$

retains a cross term in its squared norm. Since its first summand tends to zero, the total tends to zero exactly when the second does. For $x = S_\pi\lambda$, (55) gives

$$\frac{x}{1+x} \leq 1 - p_\pi(\lambda) \leq \min\{x, 1\} \leq \frac{2x}{1+x}.$$

Integration proves (58). For $h(x) = x/(1+x)$,

$$h(\delta)^2\|E_i([\delta/S_\pi, \infty))z\|^2 \leq \int h(S_\pi\lambda)^2\, d\mu_i^z \leq \delta^2\|z\|^2 + \|E_i([\delta/S_\pi, \infty))z\|^2.$$

The lower bound proves necessity. For sufficiency choose $\delta$ first and then its filter event. This proves the equivalence without interchanging an integral and a general net. □

If $z$ belongs to the finite-stage form domain, then

$$\|[I - p_\pi(A_i)]z\|^2 \leq S_\pi q_i(z).$$

Thus $S_\pi q_i(z) \to 0$ suffices but is not necessary. For example, rotate the zero eigenspace of $\mathrm{diag}(0, n^4)$ by angle $1/n$ from $e_1$. The generalized limit has $K = \mathbb{R}e_1$ and $A_* = 0$. Kernel projections converge, so every independent long-time limit preserves $e_1$; nevertheless for $S_n = n$ its energy bound is $n^5 \sin^2(1/n) \to \infty$. Small mass at very large eigenvalues can have large energy and small state effect.

For any actual response realization $D_i$ of the finite-stage form,

$$\|D_i p_\pi(A_i)f\| \leq \frac{\|f\|}{2\sqrt{S_\pi}}. \tag{59}$$

This follows from the last scalar bound in (55) and also establishes membership in the response domain. The constant is sharp for one step $S$ and a scalar operator $A = 1/S$. No response orientation condition is needed for convergence to zero; this cannot replace stationary state survival. Strongly varying inputs are allowed by contraction and (59). Strong damping is not norm-uniform: $Ae_n = n^{-1}e_n$ on $\ell^2$ has zero kernel but $\|(I + SA)^{-1}\| = 1$ for every $S$.

**Comparison and scope.** The preceding characterizations use standard polar decomposition and spectral calculus. Actual strong convergence of gradient derivations is already present in the metric-measure theorem of Ambrosio–Stra–Trevisan [2, Theorem 6.4]; graph-monotone polar continuity occurs in Accardi–Longo [1, Section 3, displayed Theorem 1]. The additional object isolated here is the necessary and sufficient active orientation, with adjoint recovery only on $\overline{\mathrm{ran}\, D}$. For damping, the new formulation separates convergence of the zero spectral atoms from survival at a prescribed moving scale. Generalized compact-time semigroup convergence is established form theory [4]. These comparisons do not certify global priority of the present equivalence statements.

FINER records the independent or coupled filters and the exact response data. The analytic hypotheses are indispensable; the name or an infinitesimal germ supplies none of them. Theorems here make no use of unresolved continuum field-theory assertions.

**Corollary 11.3** (Retained conditional monotone damping)**.** *For the conditional completion of Theorem 7.1, let $Z = \ker A_*$. For every $f \in H$, under independent conditional refinement and $S_\pi \to \infty$,*

$$p_\pi(A_i)f \to P_Z f, \qquad D_i p_\pi(A_i)f \to 0, \qquad \|D_i p_\pi(A_i)f\| \le \frac{\|f - P_Z f\|}{2\sqrt{S_\pi}}.$$

*No upper-mesh restriction is required. For the converse, let $(i, \pi)$ vary along any proper filter whose conditional index tends to its refinement tail, without presupposing a time limit. If $P_K f \notin Z$, convergence of the displayed states to $P_Z f$ is equivalent to $S_\pi \to \infty$ along this filter.*

*Proof.* For $z \in Z$, monotonicity gives $0 \le e_i(z) \le Q(z) = 0$. Since $z \in \operatorname{Dom} Q \subset V$, form Cauchy–Schwarz gives $e_i(z, v) = 0$ for every $v \in V$. The representation of the closed form therefore places $z$ in $\operatorname{Dom} A_i$ with $A_i z = 0$: the limiting kernel is inherited exactly. Theorem 11.1 gives the state limit; the spectral response bound applies to $f - P_Z f$ because all finite responses vanish on $Z$. For necessity, if cumulative time fails to tend to infinity, some $M > 0$ bounds it frequently. With $f_0 = f - P_Z f$, the lower scalar bound gives on that frequent set

$$\|p_\pi(A_i)f - P_Z f\|^2 \ge \langle f_0, e^{-2MA_i} f_0\rangle \longrightarrow \langle P_K f_0, e^{-2MA_*} P_K f_0\rangle > 0.$$

The final strict inequality follows from $P_K f_0 \neq 0$ and strict positivity of the exponential multiplier at finite spectral values. Intersecting the frequent set with a suitable refinement event contradicts state convergence. This proves the original surviving-component converse without a uniform spectral gap. □

## 12 Conditional recovery for a perturbed product energy

Let $\Omega = \{-1, 1\}^{\mathbb{N}}$ with the fair Bernoulli product measure $\mu = (\frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1)^{\otimes\mathbb{N}}$. For a finite set $S \subset \mathbb{N}$ put $w_S(\omega) = \prod_{k\in S} \omega_k$. The $w_S$ form an orthonormal basis of $L^2(\Omega)$, including $w_\varnothing = 1$. Choose finite numbers $\lambda_S \ge 0$ with $\lambda_\varnothing = 0$, allowing $\sup_S \lambda_S = \infty$, and set

$$q_0(u) = \sum_S \lambda_S |\langle u, w_S\rangle|^2, \qquad V = \{u : q_0(u) < \infty\}.$$

This is a densely defined closed form by the weighted $\ell^2$ representation. Let $P_n$ condition on the first $n$ coordinates. Then $P_n w_S = w_S$ when $S \subset \{1, \dots, n\}$ and zero otherwise. Consequently $P_n V \subset V$, $q_0(P_n u) \le q_0(u)$, and $P_n u \to u$ in the graph norm of $q_0$ for every $u \in V$, by the tails of the convergent nonnegative series.

Now let $B \ge 0$ be any bounded self-adjoint operator on $H$ with $B1 = 0$, and put $q_B(u) = q_0(u) + \langle Bu, u\rangle$ on $V$. No commutation of $B$ and $P_n$ is assumed. This is a closed nonnegative form with the same domain. Let $R_{n,z}$ be its $a_{B,z}$-orthogonal projection onto $P_n V$, where $a_{B,z} = a_0 + \langle B\,\cdot, \cdot\rangle + z\langle \cdot, \cdot\rangle$. Proposition 2.3 and the diagonal orthogonality for $q_0$ give

$$\|R_{n,z}u - P_n u\|_{a_{B,z}} \le \frac{\|B\|}{\sqrt{z}} \|(I - P_n)u\|. \tag{60}$$

Indeed the cross functional is $v \mapsto \langle B(I - P_n)u, v\rangle$, and $\|v\| \le z^{-1/2}\|v\|_{a_{B,z}}$. The squared state norm on the right is exactly $\mathbb{E}[\mathrm{Var}(u \mid \mathcal{F}_n)]$. Thus conditional information loss controls the energetic correction with a coefficient independent of the number of observed coordinates. For every $u \in V$ and fixed $z > 0$, both $P_n u$ and $R_{n,z}u$ converge to $u$ in the form graph norm.

This conclusion follows from the explicit graph-tail computation and the proved correction estimate, not from martingale convergence alone.

The example includes genuinely noncommuting forms. Choose two distinct nonempty finite sets $S, T$ separated by one of the filtrations and take $B = b \otimes b$ with $b = w_S + w_T$. Then $B \geq 0$, $B1 = 0$, and $BP_n \neq P_n B$. For $u = w_T$ and an $n$ with $P_n w_T = 0$ and $P_n w_S = w_S$, the cross functional in (5) is nonzero on $w_S$, so the conditional and energy projections differ. Their discrepancy is nevertheless controlled by (60).

## 13 Conforming numerical realization

This application turns the tolerance in Theorem 7.2 into an explicit discretization condition. It uses the Neumann energy $q(v) = \int_0^1 |v'|^2$ on $H^1(0,1)$ and any finite collection of orthogonal observations on $L^2(0,1)$. Set $c_{F,t} = t^{-1} \sum_{j \in F} w_j$ and $C_{F,t} = \sum_{j \in F} w_j P_j B_t^2 P_j$. This is bounded and nonnegative with norm at most $c_{F,t}$. The operator of $e_{F,t}$ is $A + C_{F,t}$ on the Neumann operator domain $H^2$ with zero endpoint derivatives. Bounded perturbation preserves that domain; this also follows directly from the weak equation.

**Proposition 13.1** (A conditional finite-element recovery budget)**.** *Let $t, z > 0$, $u \in V_*$, and $x = z(A + C_{F,t} + z)^{-1} u$. Let $x_h$ minimize the same anchored functional over continuous piecewise linear functions on a partition with largest interval length $h$. With exact form evaluation,*

$$J_{F,t,z}(x_h) - J_{F,t,z}(x) \leq [h^2 + (z + c_{F,t})h^4] \|x''\|_2^2,$$
$$\|x''\|_2 \leq (2z + c_{F,t}) \|u\|_2.$$

*Therefore, choosing the mesh so that the right side tends to zero produces the simultaneous state and regularized conditional-response recovery of Theorem 7.2. The observation and smoothing parameters may be chosen first; the mesh is then refined until this explicit tolerance holds.*

*Proof.* The weak equation gives $Ax = z(u - x) - C_{F,t} x$. Since $\|x\|_2 \leq \|u\|_2$, the second bound follows. Let $I_h x$ be nodal interpolation. On an interval of length $\ell$, its derivative is the average of $x'$. The elementary one-dimensional Poincare estimate gives $\|(x - I_h x)'\|_{L^2} \leq \ell \|x''\|_{L^2}$. Since $x - I_h x$ vanishes at both endpoints, applying the same estimate once more gives $\|x - I_h x\|_{L^2} \leq \ell^2 \|x''\|_{L^2}$. Summing over intervals gives the two global bounds with $h$. The exact objective identity and minimizing property yield

$$J(x_h) - J(x) \leq J(I_h x) - J(x) = e_{F,t}(I_h x - x) + z \|I_h x - x\|_2^2,$$

which is bounded by the displayed expression. The approximate-minimizer part of Theorem 7.2 finishes the proof. □

This is a conforming Galerkin argument in the tradition of Cea [7]. Its role is to retain the cost of all specified conditional channels in the mesh condition. A separate quadrature or linear-solver error must be included in the objective tolerance; it cannot be omitted by calling the method conforming. Fixed mesh size is not uniform in $t \downarrow 0$ or in an expanding observation set. The argument proves a constructive finite-dimensional realization, not a new general finite-element convergence theorem.

## 14 From conditional recovery to uniform response control

Recovery of each prescribed vector is not a uniform estimate on the unit ball. The following criterion identifies the missing estimate without assuming an operator bound or a spectral gap. It is a closed-operator extension argument; its purpose here is to distinguish the information carried by all individual observations from that carried by their complete covariance matrices.

**Theorem 14.1** (Complete observation testing)**.** *Let $H, Y$ be real Hilbert spaces, let $B : \operatorname{Dom} B \subset H \to Y$ be closed and linear, and let $(v_j)_{j\in J} \subset \operatorname{Dom} B$ have dense linear span in $H$. For $C \geq 0$, the following are equivalent:*

1. *For every finite $F \subset J$ and every $(a_j)_{j\in F} \in \mathbb{R}^F$,*
$$\left\| \sum_{j\in F} a_j B v_j \right\| \leq C \left\| \sum_{j\in F} a_j v_j \right\|.$$
2. *$\operatorname{Dom} B = H$ and $\|Bx\| \leq C\|x\|$ for every $x \in H$.*

*If $Y = H$ and $S : \operatorname{Dom} S \subset H \to H$ is linear with*
$$(u, v) \in \operatorname{graph} S \quad \Longleftrightarrow \quad (v, u) \in \operatorname{graph} B,$$
*these conditions are also equivalent to $\|u\| \leq C\|Su\|$ for every $u \in \operatorname{Dom} S$. No graph-core assumption on the observations is required.*

*Proof.* On $M = \operatorname{span}\{v_j : j \in J\}$, the first inequality bounds the restriction of the already defined linear operator $B$. For $x \in H$, choose $x_n \in M$ with $x_n \to x$. The estimate makes $Bx_n$ Cauchy; completeness of $Y$ gives a limit $y$. Closedness gives $x \in \operatorname{Dom} B$ and $Bx = y$. Passing to the limit proves the bound with the same $C$. The converse is its restriction to finite sums. For the inverse-graph assertion, each $u \in \operatorname{Dom} S$ gives $Su \in \operatorname{Dom} B$ and $BSu = u$, and each $v \in \operatorname{Dom} B$ gives $Bv \in \operatorname{Dom} S$ and $SBv = v$. Substitute these identities in the respective estimates. This proves both directions, including $C = 0$; in that case the conditions can hold only in the corresponding degenerate inverse situation. □

Equivalently, every finite matrix
$$\Big(C^2\langle v_j, v_k\rangle - \langle Bv_j, Bv_k\rangle\Big)_{j,k\in F}$$
is positive semidefinite. Its off-diagonal entries cannot be omitted. The criterion remains valid for an uncountable observation family. It is classical bounded extension and closedness, not a new convergence principle. Applied to $S = A_0^{1/2}$ and its ordinary inverse $B = A_0^{-1/2}$ on the kernel-orthogonal space, it gives, for $C > 0$, exactly $q(u) \geq C^{-2}\|u\|^2$. The inverse is defined on $\operatorname{Ran} A_0^{1/2}$, not on the kernel, and its domain is not assumed to be all of $H_0$.

For a probability space the centered indicators span a dense subspace of $H_0 = L^2 \ominus \mathbb{R}1$: approximate by simple functions in $L^2$ and subtract their means. If every centered indicator is in the inverse domain, the theorem therefore applies. It characterizes uniform inverse response through finite, independently stated tests. Qualitative convergence of each test, with constants depending on the test, does not verify the uniform matrix inequality.

**Corollary 14.2** (Uniform individual tests still do not control the response)**.** *For every $\delta > 0$, one and the same Lebesgue form from the construction of Theorem 5.2 can be chosen so that, for every nontrivial measurable event $B$ and $v_B = (\mathbf{1}_B - \mu(B))/\sqrt{\mu(B)(1-\mu(B))}$,*
$$q(v_B) \leq 1 + \delta, \qquad \|A_0^{-1/2} v_B\|^2 \leq 1 + \delta, \qquad q(P_B u) \leq (1+\delta)^2 q(u) \quad (u \in \operatorname{Dom} q).$$
*Nevertheless its form domain is proper, its forward response is unbounded, and its reduced form has no positive coercivity constant.*

*Proof.* Use the simultaneous disjoint harmonic blocks in the proof of Theorem 5.2, with both weighted indicator budgets at most $\delta$. The two estimates there give exactly the first two displayed inequalities for the same diagonal form and its ordinary reduced inverse square root. Multiplying them in the exact binary cost formula gives the third. The unit harmonic vectors with weights tending respectively to infinity and to zero are unchanged. They prove both unboundedness and failure of coercivity, and the weighted domain remains proper. Complete observation testing now excludes any uniform finite-combination inverse bound. □

Thus the finite matrix condition in Theorem 14.1 fails for every finite $C$. This is an explicit consequence of the established simultaneous construction, not a separate priority claim. Its distinction from testing an orthonormal basis alone is that the bounds hold for *every* measurable binary event on a fixed nonatomic space. The obstruction remains after the norm-resolvent and energetic regularization of Theorem 8.1. It therefore separates observable recovery, exact inverse-domain membership, and uniform inverse response within the actual FINER construction.

## 15 Transport of the entire conditional energetic structure

The relevant notion of transport must act on events as well as states. The following assertion is classical unitary transport with its probability structure made explicit. It introduces no locality claim.

**Proposition 15.1** (Probability and energetic transport)**.** *Let $\phi : (\Omega', \mathcal{A}', \mu') \to (\Omega, \mathcal{A}, \mu)$ be an isomorphism of probability spaces modulo null sets. Regard all sub-$\sigma$-algebras modulo completion by ambient null sets. Define $U : L^2(\mu) \to L^2(\mu')$ by $Uf = f \circ \phi$. Let $q$ be a densely defined closed nonnegative form on $L^2(\mu)$ with associated operator $A$, and set*

$$\operatorname{Dom} q' = U \operatorname{Dom} q, \qquad q'(Uu) = q(u).$$

*Then $U$ is unitary, $U1 = 1$, and*

$$UP_{\mathcal{G}}U^{-1} = P_{\phi^{-1}\mathcal{G}}, \qquad A' = UAU^{-1}, \quad \operatorname{Dom} A' = U \operatorname{Dom} A.$$

*The square-root domains, kernel-orthogonal spaces, and ordinary reduced inverse-square-root domains are transported by $U$ as well. All binary costs and finite observation Gram matrices are unchanged. For fixed weights, the completed response form and all independent recovery nets of Section 7 are transported isometrically, with the same constants and the same index filters.*

*Proof.* Change of variables gives $\|Uf\|_2 = \|f\|_2$. The inverse measure-space map gives surjectivity. The isomorphism maps the closed subspace of $\mathcal{G}$-measurable $L^2$ functions onto that of $\phi^{-1}\mathcal{G}$-measurable functions. The orthogonal-projection characterization of conditional expectation proves the first identity; all statements are about equivalence classes, so null-set modifications do not affect them.

The map $u \mapsto Uu$ is an isometry also for the form norms, proving closedness and density of $q'$. If $u \in \operatorname{Dom} q$, existence of $h$ with $q(u, v) = \langle h, v\rangle$ for every $v \in \operatorname{Dom} q$ is equivalent to $q'(Uu, w) = \langle Uh, w\rangle$ for every $w \in \operatorname{Dom} q'$. This proves the maximal operator domain and action. Conjugating the spectral resolution gives the positive square root. In particular $\operatorname{Dom} A'^{1/2} = U \operatorname{Dom} A^{1/2}$. On the kernel complements, its inverse has the reverse graph, so its domain is the image of $\operatorname{Ran} A_0^{1/2}$; this proves the inverse-domain assertion without making a generalized-inverse convention on the kernel.

Normalized centered indicators map to normalized centered indicators with the same probabilities. Inner products of all response vectors are unchanged. The componentwise unitary $\mathcal{U} = U \oplus \bigoplus_j U$ intertwines the completed response and each regularized response. Finite subsums, their suprema, and all graph norms are therefore equal. Uniqueness of the anchored minimizer transports the recovery net; isometries preserve its strong limits along the original product filter. No interchange of integration with an arbitrary net is involved. □

Every standard atomless probability space admits such a transport from Lebesgue probability, with completed spaces interpreted modulo null sets [12, Theorem A.7 and Corollary A.11]. Consequently Theorem 5.2 and Corollary 14.2 hold on each of these spaces. This extension is a consequence of the probability-space isomorphism theorem, not an additional rigidity theorem. Standard probability spaces with atoms are not thereby isomorphic to Lebesgue probability; Theorem 4.1 remains a specific atomic construction, not a claim for every atomic probability vector.

The same argument applies to completed Haar probability on an infinite compact metrizable group, or to a finite product of such groups, and to a probability measure with a strictly positive integrable density on such a space. Indeed a positive point mass in an infinite group would, by translation invariance, give infinitely many equal positive point masses, contrary to total mass one. On a standard Borel space atomlessness is equivalent to absence of point masses. Absolute continuity preserves atomlessness, while positivity preserves the null sets. These facts give the probability-space hypotheses; they do not identify the transported form with a physical energy or prove locality. Standard noncompact atomless probability spaces are also covered. Infinite measures require a different normalization, since the constant need not lie in $L^2$.

An arbitrary Hilbert unitary is insufficient. On three equal atoms, fix the constants and rotate the two-dimensional mean-zero space through an angle not preserving its three centered-indicator lines. This is a unitary but it sends a binary conditional projection to a projection that is not a binary conditional expectation on that probability space. For nonseparable atomless spaces a Lebesgue isomorphism need not exist. For a family of conditional Gibbs measures, separate isomorphisms at each parameter do not provide a jointly measurable transport or uniform energetic estimates. Those are additional hypotheses, not consequences of disintegration alone.

## 16 What refinement preserves, and what it cannot supply

The author-developed FINER construction retains a state, its energetic response, and the filters governing approximation. A vanishing scale may be represented by a filtered family or its germ. The exact germ quotient, an approximate resolution relation, and a norm estimate are different mathematical objects. Only the last supplies the constants needed in a coercivity or orientation theorem. The graph-completion and independent-parameter recovery results above explain how verified estimates propagate. The counterexamples show why those estimates cannot be inferred from the indexing language.

There is no general replacement of Gamma convergence [9], Mosco convergence, martingale convergence, or finite-element approximation. The finite-element construction remains an explicit instance of the classical variational mechanism [7]. The standard-analytic infinitesimal vocabulary does not assert a transfer principle or a nonstandard field extension; nonstandard analysis is a separate comparison [23]. Nor does the present theorem assume that conditional expectation is contractive in every norm: even mixed-norm conditional expectations require substantive hypotheses [18].

The decisive distinctions are quantitative. Exact all-binary contractivity is rigid, but

every positive uniform tolerance admits unbounded gapless energies. A finite commutator gap gives a checkable replacement; a qualitative connected observation graph alone does not give a dimension-uniform number. Scalar energies determine positive square-root responses, but an independently specified response requires active orientation. Strong resolvents control positive finite times and eliminate nonstationary components under long-time refinement, but survival of a limiting zero mode depends on the kernel projections or on the specified cumulative spectral scale.

The current results do not determine the optimal nonlinear all-binary constant for general finite atom weights, nor a sharp asymptotic constant as the number of equal atoms grows. The manuscript gives exact three-atom constants, explicit general bounds, an exact four-atom observation-gap calculation, and rigorous obstructions. It does not claim complete classification of all weighted geometries. The literature audit records exact implication classifications and access limits; absence of an identical located statement is not proof of priority.

All these conclusions are independent of continuum Yang–Mills existence, a physical mass gap, or Navier–Stokes regularity. The separate Wilson investigation may use a verified conditional recovery result, but none of its unresolved physical estimates is a premise of this article.

# A Verification and reproducibility

The confidential supplement provides the exact numbered-result inventory, source hashes, classical reductions, and manuscript-to-Lean correspondence. Theorem 4.1 and Theorem 5.2 retain their frozen statements and have complete project-defined L4 verification, including their explicit operators and domains. Lemma 6.2, the norm form of Theorem 14.1, and the explicit simultaneous bounds in Corollary 14.2 are also checked. The arbitrary observation-family graph and its maximal domain are verified supporting parts of Theorem 7.1.

The remaining general spectral-regularization, variational, polar-response, and dynamical arguments are proved in the text but are not all verified end to end in Lean. In particular, this article does not claim complete manuscript formalization. A clean Lake build, a fresh replay through the Lean kernel, and transitive axiom audits check the supplied formal results; they do not prove correspondence for unformalized assertions. Only the usual Lean foundations occur in the audited closures. Internal simulated reviews are not external peer review or evidence of publication priority.

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The author reviewed and evaluated the resulting mathematical arguments and takes full responsibility for the content, originality, and correctness of the manuscript.

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