PROJECTION-VALUED DEFECTS AND FAITHFUL POSITIVITY IN HILBERT–PÓLYA-TYPE OPERATOR FRAMEWORKS

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Abstract. This paper develops a defect-theoretic reduction of faithful positivity in Hilbert–Pólya-type operator frameworks. The point is not to give an unconditional proof of the Riemann Hypothesis, but to isolate how far the positivity problem for the genuine prime-side Weil kernel can be narrowed by operator theory before the remaining arithmetic input is required. The framework separates null-pair gauge compensation (NGC), which preserves the genuine kernel, from faithful arithmetic norm–GNS factorisation (FANG), which is the prime-side square-norm realisation

$$W_{\mathbb{P}}(f^{\sharp} * f) = \|\mathcal{A}_{\mathbb{P}} f\|^2.$$

NGC is the replacement

$$S_K \mapsto S_K + D, \qquad S_{KK} \mapsto S_{KK} + JD, \qquad D + J(JD) = 0,$$

so it redistributes mass between Hermitian and skew-Hermitian channels without changing the genuine kernel $K_{\mathbb{P}} = S_K + JS_{KK}$. The associated compensated block is unitarily equivalent to the direct sum of the faithful prime channel $K_{\mathbb{P}}$ and the auxiliary channel $S_K - JS_{KK} + 2D$. Hence the gauge may regularise an auxiliary channel, but it cannot manufacture positivity of the prime channel.

The main new contribution is the shifted-root defect hierarchy. Starting from a candidate shifted square root

$$K'^2 = K_{\mathbb{P}} - I, \qquad A = K' + J,$$

we prove

$$K_{\mathbb{P}} - AA^* = A(K' - K'^*).$$

Thus the candidate factorisation $K_{\mathbb{P}} = AA^*$ is equivalent to the vanishing of the self-adjointness defect $A(K' - K'^*)$. Writing $K' = H + S$, with $H = H^*$ and $S^* = -S$, this becomes

$$K_{\mathbb{P}} - AA^* = 2AS,$$

so the skew-Hermitian component S is the entire shifted-root obstruction. On pure-skew branches $K' = S$, writing $S = JB$ with $B = B^*$, the defect equation reduces to

$$B(B + I) = 0.$$

Equivalently, $B = -P$, where $P = P^* = P^2$ is an orthogonal projection, and then

$$K_{\mathbb{P}} = I - P = AA^* \succeq 0.$$

Thus pure-skew defects are projection-valued null-channel branches, not sources of negative spectrum. The remaining shifted-root spectral obstruction is the nonprojection subunit branch, equivalently spectrum in $(-\infty, 0) \cup (0, 1)$ in the normal spectral model, reducing to $(0, 1)$ once nonnegativity is known.

A final nilpotent refinement closes the shifted-root route conditionally. Let

$$\Delta_{K'} := K_{\mathbb{P}} - AA^*$$

be the shifted-root factorisation defect. Since $\Delta_{K'}$ is self-adjoint, confinement of $\Delta_{K'}$ to a nilpotent faithful defect ideal forces $\Delta_{K'} = 0$. Hence

$$K_{\mathbb{P}} = AA^*,$$

so FANG follows with $\mathcal{A}_{\mathbb{P}} = A^*$. On pure-skew branches this nilpotent confinement kills the polynomial defect $B(B+I)$, forcing $B = -P$ for an orthogonal projection P, and therefore excluding the residual fractional subunit branch. Thus, under faithful nilpotent shifted-root confinement and a separate CRE determinant zero-set identity for Ξ, the usual Hilbert–Pólya closure follows conditionally from off-axis coercivity of the spectral pencil. The paper is therefore a conditional structural reduction whose novelty is the narrowing of faithful positivity to nilpotent-confined projection-valued defects.

1. Introduction

Hilbert–Pólya-type arguments are attractive because they would reduce the location of zeros of the completed xi-function to the spectral reality of a self-adjoint operator. Let

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s), \qquad \Xi(t) = \xi\left(\frac{1}{2}+it\right).$$

The Riemann Hypothesis is equivalent to the assertion that every zero of the entire function $\Xi(t)$, regarded as a function of the complex variable t, is real; see, for example, [4, 8, 15, 17]. The Hilbert–Pólya philosophy seeks a self-adjoint operator whose spectral determinant realises Ξ. Related classical perspectives include de Branges spaces, the Berry–Keating model, Weil positivity, and trace-formula approaches in noncommutative geometry [1, 5, 7, 18].

The formal implication from self-adjointness to real spectrum is elementary. The difficult part is faithfulness: the operator, its positivity, and any determinant identity must be obtained from arithmetic data rather than from the zeros one wants to prove real. This paper formulates this separation in an operator-theoretic way. It introduces two pieces of terminology.

First, *null-pair gauge compensation* (NGC) is a kernel-preserving channel redistribution. In a local complex slice with imaginary unit J, a Hermitian diagonal operator D satisfying $DJ = JD$ gives

$$D + J(JD) = D - D = 0.$$

Thus replacing

$$S_K \mapsto S_K + D, \qquad S_{KK} \mapsto S_{KK} + JD$$

preserves the combined Hermitian kernel

$$K_{\mathbb{P}} = S_K + JS_{KK}.$$

This is a gauge compensation, not a positivity shift $K_{\mathbb{P}} \mapsto K_{\mathbb{P}} + \lambda I$. It is therefore faithful to the original prime channel.

Second, *faithful arithmetic norm–GNS factorisation* (FANG) is the target square-norm realisation

$$W_{\mathbb{P}}(f^{\sharp} * f) = \|\mathcal{A}_{\mathbb{P}} f\|^2,$$

where $W_{\mathbb{P}}$ is the prime-side Weil distribution tested on convolution squares and $\mathcal{A}_{\mathbb{P}}$ is required to be constructed from prime-side data, the archimedean factor, and functional-equation symmetry, not from the zero ordinates. When available, FANG gives positivity, a GNS Hilbert space, and a strongly continuous unitary translation representation whose Stone generator is self-adjoint.

The first main contribution is the precise separation between what NGC can prove and what must remain a genuine arithmetic positivity statement. If

$$A_D = S_K + D, \qquad R_D = JS_{KK} - D,$$

then the compensated block

$$C_D(K_{\mathbb{P}}) = \begin{pmatrix} A_D & JR_D \\ -JR_D & A_D \end{pmatrix}$$

is unitarily equivalent to

$$K_{\mathbb{P}} \oplus (S_K - JS_{KK} + 2D).$$

Consequently, the gauge can make the auxiliary channel positive under lower-boundedness assumptions, but positivity of the faithful prime channel is exactly the original problem. This prevents a common pitfall: a diagonal compensation may regularise an auxiliary block, but it cannot by itself prove positivity of the genuine prime Weil kernel.

The second and sharper contribution is the defect hierarchy for shifted-root attempts at FANG. If a candidate shifted square root K' satisfies

$$K'^2 = K_{\mathbb{P}} - I$$

and commutes with J, then for $A = K' + J$ one has

$$K_{\mathbb{P}} - AA^* = A(K' - K'^*).$$

Thus the candidate positive-square factorisation is controlled by a precise self-adjointness defect. Decomposing

$$K' = H + S, \qquad H = H^*, \qquad S^* = -S,$$

turns this into

$$K_{\mathbb{P}} - AA^* = 2AS.$$

The obstruction is exactly the skew-Hermitian channel. On a pure-skew branch $K' = S$, the defect equation forces $(S + J)S = 0$. Writing $S = JB$ with $B = B^*$, this becomes $B(B + I) = 0$, hence $B = -P$ for an orthogonal projection P. Consequently the branch satisfies

$$K_{\mathbb{P}} = I - P = AA^* \succeq 0.$$

Therefore pure-skew branches are not negative-spectrum obstructions; they are projection-valued null-channel branches. The unresolved spectral obstruction is pushed to the nonprojection subunit spectrum, namely $(-\infty, 0) \cup (0, 1)$ in the normal spectral model, or simply $(0, 1)$ once nonnegativity is known.

The nilpotent refinement closes this residual obstruction under a precise additional hypothesis. The shifted-root defect

$$\Delta_{K'} := K_{\mathbb{P}} - AA^*$$

is automatically self-adjoint. Therefore, if it is confined to a nilpotent operator ideal $\mathfrak{N}$, with

$$\mathfrak{N}^q = \{0\},$$

then

$$\Delta_{K'}^q = 0 \qquad \Longrightarrow \qquad \Delta_{K'} = 0.$$

Consequently $K_{\mathbb{P}} = AA^*$, and the faithful prime alignment factorisation follows with $\mathcal{A}_{\mathbb{P}} = A^*$, provided the shifted root and the nilpotent ideal

are constructed from prime-side arithmetic data. In the normal spectral model, the same nilpotent collapse removes the residual fractional branch $\mathrm{Spec}(K_{\mathbb{P}}) \cap (0,1)$.

Status Note 1.1 (Mathematical status). This article is a conditional structural reduction. The local slice algebra, the NGC identity, the compensated block diagonalisation, the Schur-complement dominance criterion, the square-root consequence of genuine positivity, the shifted-root defect identity, the projection-valued pure-skew classification, the self-adjoint nilpotent collapse, the GNS construction under positivity and strong continuity, and the off-axis coercivity of the spectral pencil are proved from stated operator-theoretic hypotheses. The arithmetic construction of a faithful shifted root, the faithful nilpotent defect ideal, and the CRE determinant zero-set identity for Ξ are not proved here. They are isolated as independent non-circular inputs.

Organisation. Section 2 records the local complex-slice and positivity conventions. Section 3 gives separate definitions of NGC and FANG. Section 4 recalls the prime-side Weil form at the schematic level needed for the operator reduction. Section 5 proves the NGC channel decomposition. Section 6 gives the Schur-complement dominance criterion, square-root factorisation, shifted-root self-adjoint defect criterion, skew-channel reduction, and projection-valued pure-skew classification. Section 7 proves the nilpotent shifted-root closure: self-adjoint nilpotent confinement forces the factorisation defect to vanish, yields FANG, and removes the residual fractional branch in the normal spectral model. Section 8 derives the GNS Hilbert–Pólya generator from FANG. Section 9 proves the off-axis coercivity identity and states the conditional Hilbert–Pólya closure. Section 10 records non-circularity requirements and limitations.

2. Preliminaries

The paper uses only a local complex slice of the AXIOM–ALH–CRE framework of [12]. The global tensor-sorted algebra is not needed for the operator-theoretic statements below.

Standing Assumption 2.1 (Local slice)**.** Let $\mathcal{H}$ be a Hilbert space over a local complex slice generated by a scalar operator J satisfying

$$J^2 = -I, \qquad J^* = -J.$$

All operators appearing in slice-linear products are assumed to commute with J on their common invariant domain. In the bounded setting this simply means $TJ = JT$.

The model example is the ordinary complex Hilbert space regarded as a real Hilbert space, with J acting as multiplication by i. The assumptions above ensure that multiplication by J converts skew-Hermitian channels into Hermitian channels.

Lemma 2.2 (Skew-to-Hermitian conversion)**.** *Let S_{KK} be skew-Hermitian and suppose $S_{KK}J = JS_{KK}$. Then JS_{KK} is Hermitian.*

Proof. Using $J^* = -J$, $S_{KK}^* = -S_{KK}$, and commutation,

$$(JS_{KK})^* = S_{KK}^* J^* = (-S_{KK})(-J) = S_{KK}J = JS_{KK}.$$

□

Definition 2.3 (CRE real block)**.** If $T = H + JK$, with H and JK Hermitian channels in the slice, its conjugation-normalised real block is

$$\Phi_{\mathrm{conj}}(T) = \begin{pmatrix} H & K \\ -K & H \end{pmatrix}.$$

This is the usual realification of a complex operator, up to a fixed orthogonal change of coordinates.

The following standard facts will be used without further comment: realification preserves products and adjoints in the local slice; positivity of a Hermitian slice operator is equivalent to positivity of its CRE real block; and for self-adjoint bounded operators the form order is the usual quadratic-form order. These are standard matrix and operator-theoretic facts; see [2, 9, 11, 13].

Convention 2.4 (Bounded and form versions)**.** Most statements are written in the bounded operator setting to avoid obscuring the main mechanism with domain notation. The corresponding unbounded statements are valid at the level of closed semibounded forms provided the common form domains, closures, relative form bounds, and invariance under J are specified. This domain control is part of the non-circularity requirements in Section 10.

3. The NGC and FANG principles

This section gives the two separate notions requested in the introduction. They are logically independent: NGC is a kernel-preserving gauge decomposition, while FANG is a faithful positivity and GNS factorisation hypothesis.

Definition 3.1 (NGC: null-pair gauge compensation)**.** Let

$$K = S_K + JS_{KK}$$

be a Hermitian slice operator, where $S_K = S_K^*$, $S_{KK}^* = -S_{KK}$, and both channels commute with J on their common domain. A *null-pair gauge compensation datum*, abbreviated NGC, is a Hermitian operator $D = D^*$, commuting with J, together with the replacement

$$S_K \longmapsto S_{K,D} := S_K + D, \qquad S_{KK} \longmapsto S_{KK,D} := S_{KK} + JD.$$

The associated null-pair identity is

$$N(D) := D + J(JD) = 0.$$

The compensation is called faithful if it preserves the genuine kernel:

$$S_{K,D} + JS_{KK,D} = S_K + JS_{KK} = K.$$

Thus NGC is not the operation $K \mapsto K + \lambda I$. It changes the channel representation of K, not K itself.

Definition 3.2 (FANG: faithful arithmetic norm–GNS factorisation)**.** Let $\mathcal{S}_{\mathrm{adm}}$ be an admissible test space, and let $W_{\mathbb{P}}$ be a Hermitian sesquilinear distributional form on convolution-pair test functions. A *faithful arithmetic norm–GNS factorisation*, abbreviated FANG, consists of a Hilbert space $\mathcal{K}_{\mathbb{P}}$ and a closable operator or transform

$$\mathcal{A}_{\mathbb{P}} : \mathcal{S}_{\mathrm{adm}} \longrightarrow \mathcal{K}_{\mathbb{P}}$$

such that

$$W_{\mathbb{P}}(g^\sharp * f) = \langle \mathcal{A}_{\mathbb{P}} f, \mathcal{A}_{\mathbb{P}} g \rangle_{\mathcal{K}_{\mathbb{P}}} \qquad (f, g \in \mathcal{S}_{\mathrm{adm}}).$$

In particular,

$$W_{\mathbb{P}}(f^\sharp * f) = \|\mathcal{A}_{\mathbb{P}} f\|^2_{\mathcal{K}_{\mathbb{P}}} \geq 0.$$

The adjective *faithful* means that $\mathcal{A}_{\mathbb{P}}$ is constructed from prime-side data, the archimedean gamma contribution, and functional-equation symmetry, and not from the zero ordinates of Ξ.

Remark 3.3 (Diagonal and polarised forms). If only the diagonal identity $W_{\mathbb{P}}(f^\sharp * f) = \|\mathcal{A}_{\mathbb{P}} f\|^2$ is established and the form is Hermitian sesquilinear on the admissible class, the polarised identity follows from the usual polarisation identity. For clarity, Definition 3.2 builds in the polarised version.

Remark 3.4 (Logical roles). NGC and FANG should not be conflated. NGC supplies a faithful decomposition and a way to isolate an auxiliary channel. FANG supplies genuine prime-channel positivity. The central point of the paper is that NGC cannot replace FANG.

4. Prime-side convolution squares

Let f be an admissible test function on $\mathbb{R}$, for example a Schwartz function satisfying the additional holomorphy and decay assumptions required by the explicit formula. Define

$$f^\sharp(x) = \overline{f(-x)}, \qquad (f^\sharp * f)(x) = \int_{\mathbb{R}} f^\sharp(y) f(x - y)\, \mathrm{d}y.$$

With the usual Fourier normalisation,

$$\widehat{f^\sharp * f}(\xi) = \left|\widehat{f}(\xi)\right|^2.$$

This is the reason convolution squares are the natural test objects for positivity; it is also the form in which positive-definite distribution criteria enter the explicit-formula setting [3, 18].

The prime-side Weil distribution for the completed xi-function may be written schematically as

$$W_{\mathbb{P}}(g) = W_{\infty}(g) + W_0(g) - \sum_{p}\sum_{m\geq 1}\frac{\log p}{p^{m/2}}\big(g(m\log p) + g(-m\log p)\big),$$

where W_{∞} is the archimedean gamma contribution and W_0 contains the pole and main-term corrections. The exact normalisation depends on Fourier-transform conventions and is not used in the block-operator mechanism; the explicit-formula background is classical [4, 5, 18].

Hypothesis 4.1 (Prime-side kernel representation)**.** Assume that on $\mathcal{S}_{\mathrm{adm}}$ the prime-side Weil form is represented by a Hermitian slice kernel $K_{\mathbb{P}}$:

$$W_{\mathbb{P}}(g^{\sharp} * f) = \langle K_{\mathbb{P}} f, g\rangle\,.$$

Assume further that, in the chosen slice,

$$K_{\mathbb{P}} = S_K + JS_{KK},$$

where $S_K = S_K^*$, $S_{KK}^* = -S_{KK}$, and both commute with J.

Under Hypothesis 4.1, FANG is equivalent to a faithful square-root realisation

$$K_{\mathbb{P}} = \mathcal{A}_{\mathbb{P}}^*\mathcal{A}_{\mathbb{P}}$$

on the admissible form domain. The word *faithful* is essential: the square root $K_{\mathbb{P}}^{1/2}$ exists abstractly once $K_{\mathbb{P}} \geq 0$, but that does not by itself construct an arithmetic transform from prime-side data.

5. NGC channel decomposition

The algebraic content of NGC is simple but important: the compensation changes the decomposition while preserving the kernel.

Proposition 5.1 (Kernel preservation under NGC)**.** *Let $D = D^*$ commute with J. With*

$$S_{K,D} = S_K + D, \qquad S_{KK,D} = S_{KK} + JD,$$

one has

$$S_{K,D}^* = S_{K,D}, \qquad S_{KK,D}^* = -S_{KK,D},$$

and

$$S_{K,D} + JS_{KK,D} = S_K + JS_{KK}.$$

Proof. The first assertion is immediate for $S_{K,D}$. Also

$$(JD)^* = D^*J^* = D(-J) = -JD,$$

so $S_{KK,D}$ is skew-Hermitian. Finally,

$$\begin{aligned} S_{K,D} + JS_{KK,D} &= S_K + D + J(S_{KK} + JD) \\ &= S_K + D + JS_{KK} + J^2D \\ &= S_K + JS_{KK}. \end{aligned}$$

□

Definition 5.2 (Compensated Hermitian channels)**.** Given an NGC datum D, set

$$A_D := S_K + D, \qquad R_D := JS_{KK} - D.$$

Both A_D and R_D are Hermitian. Define the compensated block

$$C_D(K_{\mathbb{P}}) = \begin{pmatrix} A_D & JR_D \\ -JR_D & A_D \end{pmatrix}.$$

Lemma 5.3 (Hermiticity of the compensated block)**.** *If $A_D = A_D^*$, $R_D = R_D^*$, and both commute with J, then $C_D(K_{\mathbb{P}})$ is Hermitian.*

Proof. The diagonal entries are Hermitian. Moreover

$$(JR_D)^* = R_D^* J^* = R_D(-J) = -JR_D,$$

so the lower-left entry is the adjoint of the upper-right entry. □

Theorem 5.4 (NGC spectral-channel decomposition)**.** *Let*

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} I & I \\ -J & J \end{pmatrix}.$$

Then U is unitary and

$$U^* C_D(K_{\mathbb{P}}) U = \begin{pmatrix} A_D + R_D & 0 \\ 0 & A_D - R_D \end{pmatrix}.$$

Consequently,

$$U^* C_D(K_{\mathbb{P}}) U = \begin{pmatrix} K_{\mathbb{P}} & 0 \\ 0 & S_K - JS_{KK} + 2D \end{pmatrix}.$$

Proof. The columns $(I, -J)^T/\sqrt{2}$ and $(I, J)^T/\sqrt{2}$ are orthonormal because $J^* = -J$ and $J^2 = -I$. Hence U is unitary.

For a vector u in the common domain,

$$C_D(K_{\mathbb{P}}) \begin{pmatrix} u \\ -Ju \end{pmatrix} = \begin{pmatrix} A_D u + JR_D(-Ju) \\ -JR_D u + A_D(-Ju) \end{pmatrix}.$$

Since R_D commutes with J, one has $JR_D(-Ju) = R_D u$. Since A_D commutes with J, the lower component is $-J(A_D + R_D)u$. Thus

$$C_D(K_{\mathbb{P}}) \begin{pmatrix} u \\ -Ju \end{pmatrix} = \begin{pmatrix} (A_D + R_D)u \\ -J(A_D + R_D)u \end{pmatrix}.$$

Similarly,

$$C_D(K_{\mathbb{P}}) \begin{pmatrix} u \\ Ju \end{pmatrix} = \begin{pmatrix} (A_D - R_D)u \\ J(A_D - R_D)u \end{pmatrix}.$$

This proves the block diagonalisation. Finally,

$$A_D + R_D = (S_K + D) + (JS_{KK} - D) = K_{\mathbb{P}},$$

and

$$A_D - R_D = (S_K + D) - (JS_{KK} - D) = S_K - JS_{KK} + 2D.$$

□

Corollary 5.5 (Faithful positivity equivalence)**.** *In the bounded setting,*

$$C_D(K_{\mathbb{P}}) \succeq 0$$

if and only if

$$K_{\mathbb{P}} \succeq 0 \qquad \text{and} \qquad S_K - JS_{KK} + 2D \succeq 0.$$

The same equivalence holds for strict positivity.

Proof. This is immediate from the unitary equivalence in Theorem 5.4. □

Corollary 5.6 (What NGC can and cannot do)**.** *Assume either a finite-dimensional truncation or a form setting in which $S_K - JS_{KK}$ is bounded below. Then one can choose D sufficiently large so that*

$$S_K - JS_{KK} + 2D \succeq 0.$$

For such a choice,

$$C_D(K_{\mathbb{P}}) \succeq 0 \quad \Longleftrightarrow \quad K_{\mathbb{P}} \succeq 0.$$

Thus NGC can regularise the auxiliary channel but cannot fake positivity of the faithful prime channel.

Proof. In finite dimension choose $D = \lambda I$ with

$$\lambda \geq \frac{1}{2} \max\{0, -\lambda_{\min}(S_K - JS_{KK})\}.$$

The form-bounded case is the same argument in quadratic-form order. The equivalence then follows from Corollary 5.5. □

6. Dominance, shifted-root defects, and projection-valued branches

The following criterion is a Schur-complement form of two-channel positivity. It is standard in positive operator theory and matrix analysis [2, 10, 11, 13].

Definition 6.1 (Compensated dominance)**.** Assume that A_D is strictly positive and boundedly invertible. The compensated dominance condition is

$$\left\| A_D^{-1/2} R_D A_D^{-1/2} \right\| \leq 1,$$

where $R_D = R_D^*$.

Theorem 6.2 (Dominance equals two-channel positivity)**.** *Assume $A_D \succ 0$ and $R_D = R_D^*$ are bounded operators. Then*

$$\left\| A_D^{-1/2} R_D A_D^{-1/2} \right\| \leq 1$$

if and only if

$$A_D + R_D \succeq 0 \qquad \text{and} \qquad A_D - R_D \succeq 0.$$

Equivalently, in the NGC setting,

$$\left\| A_D^{-1/2} R_D A_D^{-1/2} \right\| \leq 1$$

if and only if

$$K_{\mathbb{P}} \succeq 0 \qquad \textit{and} \qquad S_K - JS_{KK} + 2D \succeq 0.$$

Proof. Set

$$B_D = A_D^{-1/2} R_D A_D^{-1/2}.$$

Since B_D is self-adjoint, $\|B_D\| \leq 1$ is equivalent to

$$-I \preceq B_D \preceq I.$$

Conjugating by $A_D^{1/2}$ gives

$$-A_D \preceq R_D \preceq A_D,$$

which is equivalent to

$$A_D + R_D \succeq 0, \qquad A_D - R_D \succeq 0.$$

The final assertion follows from

$$A_D + R_D = K_{\mathbb{P}}, \qquad A_D - R_D = S_K - JS_{KK} + 2D.$$

□

Proposition 6.3 (Square-root consequence of genuine positivity)**.** *If $K_{\mathbb{P}} \succeq 0$ as a Hermitian operator or closed nonnegative form, then there exists a positive square-root operator $B_{\mathbb{P}} = K_{\mathbb{P}}^{1/2}$ such that*

$$K_{\mathbb{P}} = B_{\mathbb{P}}^* B_{\mathbb{P}}$$

in the appropriate operator or form sense. Consequently,

$$W_{\mathbb{P}}(f^\sharp * f) = \|B_{\mathbb{P}} f\|^2$$

for all admissible f in the form domain.

Proof. This is the standard functional calculus for positive self-adjoint operators or closed nonnegative forms; see [6, 13]. Applying it to $K_{\mathbb{P}}$ gives $B_{\mathbb{P}} = K_{\mathbb{P}}^{1/2}$, and hence

$$\langle K_{\mathbb{P}} f, f\rangle = \langle B_{\mathbb{P}} f, B_{\mathbb{P}} f\rangle = \|B_{\mathbb{P}} f\|^2 .$$

□

Lemma 6.4 (Shifted-square-root self-adjoint defect criterion)**.** *Let K' be a bounded slice-linear operator commuting with J, and assume*

$$K'^2 = K_{\mathbb{P}} - I.$$

Set

$$A := K' + J.$$

Then

$$K_{\mathbb{P}} = (K' + J)(K' - J)$$

and

$$K_{\mathbb{P}} - AA^* = A(K' - K'^*).$$

Consequently,

$$K_{\mathbb{P}} = AA^* \quad \Longleftrightarrow \quad A(K' - K'^*) = 0.$$

Since $A = K' + J$, equivalently,

$$K_{\mathbb{P}} = AA^* \quad \Longleftrightarrow \quad (K' + J)(K' - K'^*) = 0.$$

If, in addition, $A = K' + J$ is injective, then

$$K_{\mathbb{P}} = AA^* \quad \Longleftrightarrow \quad K' = K'^*.$$

The same identities hold on a common invariant core in the unbounded setting, provided all products and adjoints are defined there and the resulting identities are interpreted as form or graph-closure identities.

Proof. Since K' commutes with J and $J^2 = -I$,

$$\begin{aligned}(K' + J)(K' - J) &= K'^2 - K'J + JK' - J^2 \\ &= K'^2 + I = K_{\mathbb{P}}.\end{aligned}$$

Also

$$A^* = (K' + J)^* = K'^* - J.$$

Therefore

$$K' - J = (K'^* - J) + (K' - K'^*) = A^* + (K' - K'^*).$$

Using the first identity,

$$\begin{aligned}K_{\mathbb{P}} &= (K' + J)(K' - J) \\ &= (K' + J)\{A^* + (K' - K'^*)\} \\ &= (K' + J)A^* + (K' + J)(K' - K'^*) \\ &= AA^* + A(K' - K'^*).\end{aligned}$$

Thus

$$K_{\mathbb{P}} - AA^* = A(K' - K'^*),$$

and the equivalence $K_{\mathbb{P}} = AA^* \Longleftrightarrow A(K' - K'^*) = 0$ follows. Since $A = K' + J$, this is the same as

$$K_{\mathbb{P}} = AA^* \quad \Longleftrightarrow \quad (K' + J)(K' - K'^*) = 0.$$

If A is injective and $A(K' - K'^*) = 0$, then $(K' - K'^*)u = 0$ for every vector u, so $K' = K'^*$. The converse is immediate. □

Corollary 6.5 (Skew-channel obstruction for shifted roots)**.** *In the setting of Lemma 6.4, write the canonical Hermitian–skew decomposition*

$$K' = H + S, \qquad H := \frac{K' + K'^*}{2}, \qquad S := \frac{K' - K'^*}{2}.$$

Then $H^* = H$, $S^* = -S$, and

$$K_{\mathbb{P}} - AA^* = 2AS.$$

Consequently,

$$K_{\mathbb{P}} = AA^* \quad \Longleftrightarrow \quad AS = 0.$$

If, moreover, $A = K' + J$ is injective, then

$$K_{\mathbb{P}} = AA^* \quad \Longleftrightarrow \quad S = 0.$$

In the CRE lift, the same obstruction is the purely off-diagonal block

$$\Phi_{\mathrm{conj}}(K') - \Phi_{\mathrm{conj}}(K')^* = \begin{pmatrix} 0 & -2S \\ 2S & 0 \end{pmatrix},$$

up to the sign convention of the realification.

Proof. The definitions give $K' - K'^* = 2S$. The identity $K_{\mathbb{P}} - AA^* = A(K' - K'^*)$ from Lemma 6.4 therefore becomes $K_{\mathbb{P}} - AA^* = 2AS$, and the equivalences follow. The displayed CRE identity follows from the fact that Φ_{conj} is a $*$-isomorphism on the local slice, so $\Phi_{\mathrm{conj}}(K' - K'^*) = \Phi_{\mathrm{conj}}(K') - \Phi_{\mathrm{conj}}(K')^*$. □

Corollary 6.6 (Normal shifted roots and the kernel of the Hermitian part)**.** *Assume, in addition to the hypotheses of Corollary 6.5, that $K_{\mathbb{P}} = K_{\mathbb{P}}^*$ and*

$$K'^2 = K_{\mathbb{P}} - I.$$

Then

$$HS + SH = 0.$$

If K' is normal, then $HS = SH$, hence

$$HS = SH = 0, \qquad \operatorname{Ran} S \subseteq \operatorname{Ker} H.$$

Consequently, if $\operatorname{Ker} H = \{0\}$*, then $S = 0$, and hence*

$$K_{\mathbb{P}} = AA^*.$$

Equivalently, under normality of the shifted square-root candidate, the only place where a nonzero skew defect can survive is the kernel of the Hermitian part H.

Proof. Expanding $K'^2 = (H + S)^2$ gives

$$K'^2 = H^2 + HS + SH + S^2.$$

Since $H^2 + S^2$ is Hermitian and $K'^2 = K_{\mathbb{P}} - I$ is Hermitian, the skew-Hermitian part must vanish, so $HS + SH = 0$. Next,

$$K'K'^* = (H + S)(H - S) = H^2 - S^2 + SH - HS,$$
$$K'^*K' = (H - S)(H + S) = H^2 - S^2 + HS - SH.$$

Thus normality of K' is equivalent to $SH = HS$. Combining $HS = SH$ with $HS + SH = 0$ gives $HS = SH = 0$. Hence $H(Su) = 0$ for every vector u, so $\operatorname{Ran} S \subseteq \operatorname{Ker} H$. If $\operatorname{Ker} H = \{0\}$, then $S = 0$, and Corollary 6.5 gives $K_{\mathbb{P}} = AA^*$. □

Remark 6.7 (Spectral meaning of the kernel obstruction). The identity $K'^2 = K_{\mathbb{P}} - I$ gives, after the cancellation of the mixed term,

$$K_{\mathbb{P}} - I = H^2 + S^2.$$

Because $S^* = -S$, the form S^2 is nonpositive. Thus on $\operatorname{Ker} H$, whenever the expression is interpreted on a common invariant form domain,

$$(K_{\mathbb{P}} - I)|_{\operatorname{Ker} H} = S^2|_{\operatorname{Ker} H} \preceq 0.$$

Therefore a nonzero skew defect can hide only in a nonpositive spectral branch of $K_{\mathbb{P}} - I$ inside $\operatorname{Ker} H$. This does not prove faithful positivity, but it sharpens the remaining obstruction from an abstract self-adjointness defect to a concrete kernel/spectral-gap question for the Hermitian part of the shifted square root.

Remark 6.8 (Interpretation of the defect). Lemma 6.4 does not create positivity. Rather, it shows that a particular factorisation attempt

$$K_{\mathbb{P}} = AA^*, \qquad A = K' + J,$$

reduces to the vanishing of the self-adjointness defect of the shifted square-root candidate K'. Under injectivity of $K' + J$, the obstruction is exactly whether K' is self-adjoint. This is useful as a diagnostic inside FANG: a formal square-root candidate is faithful only after the self-adjoint defect and the prime-side construction requirements have both been controlled.

Remark 6.9 (Negative-branch interpretation of the skew defect). The decomposition $K' = H + S$ also explains when the defect should be expected to appear. The defect is not produced merely by using the CRE lift; it is the skew-Hermitian part of the shifted square root:

$$K' - K'^* = 2S.$$

Thus a purely Hermitian branch of K' has no defect, whereas a skew-Hermitian branch is entirely defect. In particular, if K' is skew-Hermitian, then

$$K'^* = -K', \qquad H = \frac{K' + K'^*}{2} = 0, \qquad S = \frac{K' - K'^*}{2} = K',$$

so $K' = S$. In a spectral-calculus picture this is exactly what occurs on a negative branch of $K_{\mathbb{P}} - I$: if an eigenvalue of $K_{\mathbb{P}} - I$ is negative, the corresponding square root is imaginary in the local slice and therefore skew-Hermitian. The skew channel S may therefore be read as the part of the shifted square root supported on the negative spectral branch of $K_{\mathbb{P}} - I$. This is only an interpretation of the obstruction, not a proof that the obstruction vanishes.

Theorem 6.10 (Projection-valued classification of pure-skew branches)**.** *Assume the hypotheses of Lemma 6.4. Suppose, on an invariant reducing subspace of the shifted-root candidate, that the branch is purely skew-Hermitian:*

$$K' = S, \qquad S^* = -S,$$

and that S commutes with the local imaginary unit J. Then the candidate factorisation condition

$$K_{\mathbb{P}} = AA^*, \qquad A = K' + J,$$

is equivalent on that subspace to

$$(S + J)S = 0.$$

Writing

$$S = JB, \qquad B := -JS,$$

one has $B = B^*$. *The pure-skew defect equation is equivalent to*

$$B(B + I) = 0.$$

Consequently, with

$$P := -B,$$

one has

$$P = P^* = P^2,$$

so P *is an orthogonal projection, and*

$$K' = -JP, \qquad K_{\mathbb{P}} = I - P, \qquad A = J(I - P).$$

In particular,

$$K_{\mathbb{P}} = AA^* = I - P \succeq 0$$

on the pure-skew branch. Conversely, any orthogonal projection P *commuting with* J, *with* $K' = -JP$, *satisfies these identities on the corresponding branch.*

Proof. On the pure-skew branch, $K' - K'^* = 2S$. Hence Lemma 6.4 gives

$$K_{\mathbb{P}} = AA^* \quad \Longleftrightarrow \quad (S + J)(2S) = 0,$$

which is equivalent to $(S + J)S = 0$.

Since S commutes with J and $J^* = -J$, the operator $B = -JS$ is Hermitian:

$$B^* = (-JS)^* = S^*(-J)^* = (-S)J = -SJ = -JS = B.$$

Substituting $S = JB$ into $(S + J)S = 0$ gives

$$(JB + J)(JB) = J(B + I)JB = J^2(B + I)B = -(B + I)B.$$

Thus $(S + J)S = 0$ is equivalent to $B(B + I) = 0$.

Set $P = -B$. Since $B = B^*$, one has $P = P^*$, and $B(B + I) = 0$ becomes

$$P^2 - P = 0.$$

Thus $P = P^* = P^2$. Moreover,

$$K' = S = JB = -JP,$$

so

$$K'^2 = (-JP)^2 = J^2P^2 = -P.$$

Since $K'^2 = K_{\mathbb{P}} - I$, it follows that

$$K_{\mathbb{P}} = I - P.$$

Finally,

$$A = K' + J = -JP + J = J(I - P),$$

and hence

$$AA^* = J(I-P)\{-J(I-P)\} = -J^2(I-P)^2 = I-P = K_{\mathbb{P}}.$$

The converse is obtained by reversing these computations. □

Corollary 6.11 (Pure-skew branches are null-channel, not negative-channel, defects)**.** *Under the hypotheses of Theorem 6.10,*

$$\mathrm{Spec}(K_{\mathbb{P}}|_{\mathcal{M}}) \subset \{0,1\}$$

on the pure-skew reducing subspace $\mathcal{M}$. More precisely, $\mathrm{Ran}\, P$ *is the zero-mode subspace of $K_{\mathbb{P}}$, while* $\mathrm{Ran}(I-P)$ *is the unit branch. Thus a nonzero pure-skew projection P does not create a negative direction for the Weil form; it creates only a null direction, which is compatible with the GNS quotient by the null space.*

Proof. This follows immediately from $K_{\mathbb{P}} = I - P$ with $P = P^* = P^2$. The eigenvalues of P are 0 and 1, so the eigenvalues of $I-P$ are 1 and 0. Hence $I - P \succeq 0$, and $\mathrm{Ker}(I-P) = \mathrm{Ran}\, P$. □

Criterion 6.12 (Remaining fractional shifted-root obstruction)**.** *In a normal spectral model for the shifted-root ansatz, the branchwise factorisation*

$$K_{\mathbb{P}} = AA^*, \qquad A = K' + J,$$

is automatic on the real branch $\lambda \geq 1$ of $K_{\mathbb{P}}$ and on the projection-valued null branch $\lambda = 0$. The pure-skew defect equation excludes all nonprojection subunit branches. Equivalently, for this shifted-root route a sufficient spectral target is

$$\mathrm{Spec}(K_{\mathbb{P}}) \subset \{0\} \cup [1,\infty).$$

The unresolved spectral obstruction is therefore

$$\mathrm{Spec}(K_{\mathbb{P}}) \cap \big((-\infty,0) \cup (0,1)\big).$$

If independent arithmetic input already gives $K_{\mathbb{P}} \succeq 0$, this residual obstruction reduces to the fractional branch

$$\mathrm{Spec}(K_{\mathbb{P}}) \cap (0,1).$$

Proof. On a scalar spectral branch $K_{\mathbb{P}} = \lambda I$, the equation $K'^2 = K_{\mathbb{P}} - I$ reads $K'^2 = (\lambda - 1)I$. If $\lambda \geq 1$, choose the Hermitian branch $K' = \sqrt{\lambda - 1}\, I$. Then $K' = K'^*$, and Lemma 6.4 gives $K_{\mathbb{P}} = AA^*$.

If $\lambda = 0$, choose the pure-skew projection branch $K' = -J$. Then $A = K' + J = 0$, hence $AA^* = 0 = K_{\mathbb{P}}$. This is the rank-one scalar version of Theorem 6.10.

If $\lambda < 1$ and the branch is pure skew, write $K' = JB$ with scalar self-adjoint parameter $B = bI$ and $b^2 = 1 - \lambda$, up to the chosen sign. The defect equation requires $b(b+1) = 0$, hence $b = 0$ or $b = -1$, which gives respectively $\lambda = 1$ or $\lambda = 0$. Thus all nonprojection subunit branches are incompatible with the shifted-root candidate factorisation. This gives the stated spectral target and obstruction set. □

Remark 6.13 (Projection-valued defect hierarchy). The shifted-root analysis narrows the faithful-positivity question through the hierarchy

$$K_{\mathbb{P}} - AA^* = A(K' - K'^*) \quad \rightsquigarrow \quad K' - K'^* = 2S \quad \rightsquigarrow \quad S = JB \quad \rightsquigarrow \quad B = -P, \quad P = P^* = P^2.$$

Thus the pure-skew part of the obstruction is not arbitrary. It is projection-valued and produces $K_{\mathbb{P}} = I - P$, hence a positive semidefinite null-channel branch. Before imposing the nilpotent confinement hypothesis of Section 7, what remains to be excluded for the shifted-root route is not this projection branch, but the nonprojection fractional/subunit spectrum described in Criterion 6.12.

Theorem 6.14 (FANG from compensated dominance, conditional form)**.** *Suppose the prime kernel admits an NGC datum D such that $A_D \succ 0$ and the compensated dominance condition holds. Then $K_{\mathbb{P}} \succeq 0$. Hence an abstract square-norm factorisation exists:*

$$W_{\mathbb{P}}(f^\sharp * f) = \left\| K_{\mathbb{P}}^{1/2} f \right\|^2.$$

This becomes a faithful FANG factorisation precisely when the resulting square factor, or another equivalent factor $\mathcal{A}_{\mathbb{P}}$, is constructed non-circularly from prime-side arithmetic data.

Proof. By Theorem 6.2, compensated dominance implies $K_{\mathbb{P}} \succeq 0$. The square-root factorisation then follows from Proposition 6.3. The last sentence is not an additional operator-theoretic conclusion; it records the faithfulness requirement in Definition 3.2. □

Remark 6.15 (Why the last condition matters). The implication $K_{\mathbb{P}} \succeq 0 \Rightarrow K_{\mathbb{P}} = K_{\mathbb{P}}^{1/2} K_{\mathbb{P}}^{1/2}$ is formal. It is not by itself an arithmetic construction of $\mathcal{A}_{\mathbb{P}}$. A non-circular Hilbert–Pólya programme must verify dominance or construct $\mathcal{A}_{\mathbb{P}}$ from the prime side, the archimedean factor, and the functional equation, not from the zero ordinates.

7. Nilpotent shifted-root closure

The preceding section isolates the shifted-root obstruction

$$K_{\mathbb{P}} - AA^* = A(K' - K'^*).$$

We now record the nilpotent refinement. The important point is that the nilpotent object is not the gauge D. If $D = D^*$ and D were nilpotent, then $D = 0$. The useful nilpotent object is instead the self-adjoint factorisation defect

$$\Delta_{K'} := K_{\mathbb{P}} - AA^*.$$

Throughout this section we work in the bounded slice-linear setting. The corresponding unbounded form version requires a common invariant core, closure of all products, and compatibility of the nilpotent ideal with the relevant form domains.

Definition 7.1 (Shifted-root factorisation defect)**.** Let K' be a shifted-root datum satisfying

$$K'^2 = K_{\mathbb{P}} - I, \qquad A := K' + J.$$

The shifted-root factorisation defect is

$$\Delta_{K'} := K_{\mathbb{P}} - AA^*.$$

Since both $K_{\mathbb{P}}$ and AA^* are Hermitian, one has

$$\Delta_{K'}^* = \Delta_{K'}.$$

By Lemma 6.4,

$$\Delta_{K'} = A(K' - K'^*).$$

If $K' = H + S$, with $H = H^*$ and $S^* = -S$, then

$$\Delta_{K'} = 2AS.$$

Lemma 7.2 (Self-adjoint nilpotent collapse)**.** *Let $T = T^*$ be a bounded operator. If*

$$T^q = 0$$

for some $q \geq 1$, then $T = 0$.

Proof. By the spectral mapping theorem,

$$\mathrm{Spec}(T)^q = \mathrm{Spec}(T^q) = \{0\}.$$

Hence $\mathrm{Spec}(T) = \{0\}$. Since T is self-adjoint,

$$\|T\| = r(T) = 0,$$

so $T = 0$. □

Hypothesis 7.3 (Faithful nilpotent shifted-root confinement)**.** There exists a shifted-root datum K', constructed from prime-side data, the archimedean contribution, and functional-equation symmetry, together with a defect ideal $\mathfrak{N}$, constructed from the prime-side channel algebra, the null-pair gauge structure, the shifted-root datum, and the same functional-equation symmetry, such that

$$\mathfrak{N}^q = \{0\}$$

for some $q \geq 1$, and

$$\Delta_{K'} \in \mathfrak{N}.$$

The ideal $\mathfrak{N}$ is required to be specified before the closure argument and may not be defined post hoc by imposing $\Delta_{K'} = 0$ or by using the zero ordinates of Ξ.

Theorem 7.4 (FANG from nilpotent shifted-root confinement)**.** *Assume Hypothesis 7.3. Then*

$$\Delta_{K'} = 0.$$

Consequently,

$$K_{\mathbb{P}} = AA^*.$$

With

$$\mathcal{A}_{\mathbb{P}} := A^*,$$

one has

$$K_{\mathbb{P}} = \mathcal{A}_{\mathbb{P}}^* \mathcal{A}_{\mathbb{P}}.$$

Hence FANG follows with $\mathcal{A}_{\mathbb{P}} = A^*$.

Proof. Since $\Delta_{K'} \in \mathfrak{N}$ and $\mathfrak{N}^q = \{0\}$, one has $(\Delta_{K'})^q = 0$. Because $\Delta_{K'}$ is self-adjoint, Lemma 7.2 gives $\Delta_{K'} = 0$. Therefore $K_{\mathbb{P}} = AA^*$, and setting $\mathcal{A}_{\mathbb{P}} = A^*$ gives the FANG factorisation. □

Theorem 7.5 (Pure-skew polynomial nilpotent closure)**.** *Assume that on a reducing subspace the shifted-root datum is purely skew-Hermitian:*

$$K' = JB, \qquad B = B^*, \qquad BJ = JB.$$

Then

$$K_{\mathbb{P}} = I - B^2, \qquad A = J(B + I),$$

and the shifted-root defect is

$$\Delta_{K'} = -2B(B + I).$$

If $\Delta_{K'}$ *is nilpotently confined, then* $B(B + I) = 0$. *Consequently, with* $P := -B$, *one has*

$$P = P^* = P^2,$$

and

$$K' = -JP, \qquad K_{\mathbb{P}} = I - P, \qquad A = J(I - P).$$

In particular,

$$K_{\mathbb{P}} = AA^* = I - P \succeq 0.$$

Proof. Since $K' = JB$ and $J^2 = -I$, one has $K'^2 = -B^2$, hence $K_{\mathbb{P}} = I - B^2$. Also $A = J(B + I)$. A direct computation gives

$$AA^* = (B + I)^2,$$

so

$$\Delta_{K'} = K_{\mathbb{P}} - AA^* = I - B^2 - (B + I)^2 = -2B(B + I).$$

If $\Delta_{K'} \in \mathfrak{N}$ and $\mathfrak{N}^q = \{0\}$, then

$$(\Delta_{K'})^q = 0.$$

Since $\Delta_{K'}$ is self-adjoint, Lemma 7.2 gives

$$\Delta_{K'} = 0.$$

Therefore

$$-2B(B + I) = \Delta_{K'} = 0,$$

and hence

$$B(B + I) = 0.$$

Setting $P = -B$ yields $P = P^* = P^2$, and substitution gives the remaining identities. □

Theorem 7.6 (Nilpotent exclusion of the residual fractional branch)**.** *Assume the normal spectral model of Criterion 6.12. If the shifted-root defect is faithfully nilpotently confined, then*

$$\mathrm{Spec}(K_{\mathbb{P}}) \cap \big((-\infty, 0) \cup (0, 1)\big) = \varnothing.$$

Equivalently,

$$\mathrm{Spec}(K_{\mathbb{P}}) \subset \{0\} \cup [1, \infty).$$

Proof. In the normal spectral model, every branch with $\lambda < 1$ is represented by a pure-skew shifted root on the corresponding reducing subspace:

$$K' = JB, \qquad B = B^*.$$

From Theorem 7.5 one has

$$\Delta_{K'} = -2B(B + I).$$

Since the shifted-root defect is assumed to be faithfully nilpotently confined,

$$\Delta_{K'} \in \mathfrak{N}, \qquad \mathfrak{N}^q = \{0\},$$

one has

$$(\Delta_{K'})^q = 0.$$

Since $\Delta_{K'}$ is self-adjoint, Lemma 7.2 gives

$$\Delta_{K'} = 0.$$

Hence

$$B(B + I) = 0.$$

It follows that

$$B = -P, \qquad P = P^* = P^2,$$

so

$$K_{\mathbb{P}} = I - P.$$

Therefore the pure-skew branch has spectrum contained in

$$\{0, 1\}.$$

No spectral value in

$$(-\infty, 0) \cup (0, 1)$$

can occur on such a branch. Since the real branch $\lambda \geq 1$ is already admissible, one obtains

$$\mathrm{Spec}(K_{\mathbb{P}}) \subset \{0\} \cup [1, \infty).$$

This is equivalent to

$$\mathrm{Spec}(K_{\mathbb{P}}) \cap \big((-\infty, 0) \cup (0, 1)\big) = \varnothing.$$

□

Corollary 7.7 (Closed shifted-root route to faithful positivity)**.** *Assume a faithful shifted-root datum, faithful nilpotent confinement, and the normal spectral model. Then*

$$K_{\mathbb{P}} = AA^* \succeq 0,$$

FANG holds with $\mathcal{A}_{\mathbb{P}} = A^*$*, and*

$$\operatorname{Spec}(K_{\mathbb{P}}) \cap \big((-\infty, 0) \cup (0, 1)\big) = \varnothing.$$

Proof. Combine Theorem 7.4 and Theorem 7.6. □

Remark 7.8 (Nilpotent closure hierarchy). The shifted-root route closes through the hierarchy

$$K_{\mathbb{P}} - AA^* = A(K' - K'^*) \rightsquigarrow \Delta_{K'} \in \mathfrak{N} \rightsquigarrow \Delta_{K'} = 0 \rightsquigarrow K_{\mathbb{P}} = AA^* \rightsquigarrow K_{\mathbb{P}} = \mathcal{A}_{\mathbb{P}}^* \mathcal{A}_{\mathbb{P}}.$$

The first step identifies the shifted-root obstruction as the self-adjoint factorisation defect

$$\Delta_{K'} := K_{\mathbb{P}} - AA^*.$$

Under the faithful nilpotent confinement hypothesis,

$$\Delta_{K'} \in \mathfrak{N}, \qquad \mathfrak{N}^q = \{0\},$$

one obtains

$$(\Delta_{K'})^q = 0.$$

Since $\Delta_{K'}$ is self-adjoint, Lemma 7.2 yields

$$\Delta_{K'} = 0.$$

Consequently,

$$K_{\mathbb{P}} = AA^*.$$

Defining

$$\mathcal{A}_{\mathbb{P}} := A^*,$$

one obtains the faithful square-norm factorisation

$$K_{\mathbb{P}} = \mathcal{A}_{\mathbb{P}}^* \mathcal{A}_{\mathbb{P}},$$

and therefore

$$W_{\mathbb{P}}(f^\sharp * f) = \|\mathcal{A}_{\mathbb{P}} f\|^2.$$

Thus FANG is no longer introduced as a primitive positivity assumption within the shifted-root route. Instead, positivity becomes the consequence of two independent arithmetic inputs:

faithful shifted-root construction + faithful nilpotent defect confinement.

Accordingly, the residual fractional branch is no longer treated as an independent positivity problem. Under the nilpotent refinement, it is absorbed into the faithful defect-confinement problem.

Remark 7.9 (Nilpotent closure status). The nilpotent refinement does not prove positivity unconditionally. Its role is to isolate positivity into two faithful arithmetic targets: the construction of a shifted-root datum and the construction of a nilpotent defect ideal. Under these inputs, the factorisation defect vanishes and the residual fractional branch is excluded.

8. GNS construction from FANG

FANG converts the prime-side distribution into a Hilbert space by the usual GNS procedure. This is standard functional analysis, but it is useful to record the precise conditional statement.

Hypothesis 8.1 (FANG hypothesis for the prime form)**.** Assume that $W_{\mathbb{P}}$ satisfies the FANG condition of Definition 3.2. Thus there is a faithful arithmetic transform $\mathcal{A}_{\mathbb{P}}$ such that

$$W_{\mathbb{P}}(g^{\sharp} * f) = \langle \mathcal{A}_{\mathbb{P}} f, \mathcal{A}_{\mathbb{P}} g \rangle .$$

Definition 8.2 (Prime alignment inner product)**.** Under Hypothesis 8.1, define

$$\langle f, g \rangle_{\mathbb{P}} := W_{\mathbb{P}}(g^{\sharp} * f).$$

Let

$$\mathcal{N}_{\mathbb{P}} = \{ f \in \mathcal{S}_{\mathrm{adm}} : \langle f, f \rangle_{\mathbb{P}} = 0 \}.$$

The prime-alignment Hilbert space is the completion

$$\mathcal{H}_{\mathrm{HP}} := \overline{\mathcal{S}_{\mathrm{adm}}/\mathcal{N}_{\mathbb{P}}}^{\langle \cdot, \cdot \rangle_{\mathbb{P}}}.$$

Lemma 8.3 (Unitary translation representation)**.** *Let $(U_a f)(x) = f(x - a)$. Assume the admissible class is translation-invariant and that*

$$(U_a g)^{\sharp} * (U_a f) = g^{\sharp} * f$$

in the argument of $W_{\mathbb{P}}$. Then U_a descends to a unitary operator on $\mathcal{H}_{\mathrm{HP}}$.

Proof. By translation invariance in the paired convolution square,

$$\langle U_a f, U_a g \rangle_{\mathbb{P}} = W_{\mathbb{P}}((U_a g)^{\sharp} * (U_a f)) = W_{\mathbb{P}}(g^{\sharp} * f) = \langle f, g \rangle_{\mathbb{P}} .$$

Thus U_a is isometric on the quotient and extends to a unitary operator on the completion. The group law is inherited from translations. □

Theorem 8.4 (GNS Hilbert–Pólya generator)**.** *Assume FANG and assume that the induced translation representation $a \mapsto U_a$ on $\mathcal{H}_{\mathrm{HP}}$ is strongly continuous. Then there exists a self-adjoint operator H_{HP} on $\mathcal{H}_{\mathrm{HP}}$ such that*

$$U_a = e^{-iaH_{\mathrm{HP}}} \qquad (a \in \mathbb{R}).$$

Proof. The representation is a strongly continuous one-parameter unitary group. Stone's theorem gives a self-adjoint generator H_{HP} satisfying the asserted formula [6, 14, 16]. □

9. Spectral pencil and conditional closure

This section records the spectral part of the conditional Hilbert–Pólya mechanism. It is independent of the arithmetic construction of FANG.

Definition 9.1 (Critical-line parameter)**.** Fix the same local imaginary unit J. Write

$$t = \tau + J\eta, \qquad \tau, \eta \in \mathbb{R}.$$

Under $J \leftrightarrow i$, this is the usual complex parameter $t = \tau + i\eta$. The real t-axis is the locus $\eta = 0$.

Definition 9.2 (Axis-twisted block and HP–CRE pencil)**.** Let

$$\Sigma_J = \begin{pmatrix} 0 & -J \\ J & 0 \end{pmatrix}.$$

For $t = \tau + J\eta$, set

$$C_J(t) := J\eta I_2 - \tau \Sigma_J.$$

Given a self-adjoint operator H_{HP}, define

$$H_{\mathrm{CRE}} := -H_{\mathrm{HP}} \otimes \Sigma_J,$$

and

$$P_{\mathrm{CRE}}(t) := C_J(t) \otimes I_{\mathcal{H}} - H_{\mathrm{CRE}}.$$

Equivalently,

$$P_{\mathrm{CRE}}(t) = J\eta I + (H_{\mathrm{HP}} - \tau I) \otimes \Sigma_J.$$

Lemma 9.3 (Hermitian detector)**.** *The block $C_J(t)$ is Hermitian if and only if $\eta = 0$, equivalently if and only if $t \in \mathbb{R}$.*

Proof. Since $J^* = -J$ and $\Sigma_J^* = \Sigma_J$,

$$C_J(t)^* = -J\eta I_2 - \tau \Sigma_J.$$

Equality with $C_J(t) = J\eta I_2 - \tau \Sigma_J$ holds exactly when $J\eta I_2 = 0$, hence when $\eta = 0$. □

Lemma 9.4 (Off-axis coercivity)**.** *For $t = \tau + J\eta$,*

$$P_{\mathrm{CRE}}(t)^* P_{\mathrm{CRE}}(t) = \big((H_{\mathrm{HP}} - \tau I)^2 + \eta^2 I\big) \otimes I_2.$$

In particular, if $\eta \neq 0$, then $P_{\mathrm{CRE}}(t)^* P_{\mathrm{CRE}}(t)$ is strictly positive. Hence $\operatorname{Ker} P_{\mathrm{CRE}}(t) = \{0\}$.

Proof. Set

$$A_\tau = (H_{\mathrm{HP}} - \tau I) \otimes \Sigma_J.$$

Then $A_\tau^* = A_\tau$, $(J\eta I)^* = -J\eta I$, and $J\eta I$ commutes with A_τ. Therefore

$$P_{\mathrm{CRE}}(t)^* P_{\mathrm{CRE}}(t) = (-J\eta I + A_\tau)(J\eta I + A_\tau) = \eta^2 I + A_\tau^2.$$

Since $J^2 = -I$ and $\Sigma_J^2 = I_2$,

$$A_\tau^2 = (H_{\mathrm{HP}} - \tau I)^2 \otimes I_2.$$

Thus the identity follows. If $\eta \neq 0$, then for every nonzero vector v,

$$\langle P_{\mathrm{CRE}}(t)^* P_{\mathrm{CRE}}(t) v, v \rangle = \|((H_{\mathrm{HP}} - \tau I) \otimes I_2)v\|^2 + \eta^2 \|v\|^2 > 0.$$

□

Hypothesis 9.5 (CRE determinant zero-set identity)**.** Assume that the HP–CRE pencil satisfies a determinant zero-set identity of the form

$$|\Xi(t)|^2 = E(t,\bar{t})\,\det_{\zeta}\big(P_{\mathrm{CRE}}(t)^* P_{\mathrm{CRE}}(t)\big),$$

where $E(t,\bar{t}) \neq 0$ on the domain under consideration. The identity is required to be proved by trace-formula or explicit-formula comparison, not by defining H_{HP} to have the Riemann zeros as spectrum.

Theorem 9.6 (Conditional Hilbert–Pólya closure)**.** *Assume:*

(i) the FANG hypothesis holds for the genuine prime-side Weil kernel;
(ii) the induced translation representation is strongly continuous, so that the GNS construction gives a self-adjoint H_{HP};
(iii) the CRE determinant zero-set identity of Hypothesis 9.5 holds.

Then every zero of $\Xi(t)$ is real. Equivalently, the Riemann Hypothesis follows under these hypotheses.

Proof. Write $t = \tau + J\eta$. If $\Xi(t) = 0$, then by the determinant zero-set identity and the nonvanishing of E, the determinant factor must vanish. If $\eta \neq 0$, Lemma 9.4 gives

$$P_{\mathrm{CRE}}(t)^* P_{\mathrm{CRE}}(t) = \big((H_{\mathrm{HP}} - \tau I)^2 + \eta^2 I\big) \otimes I_2 \succ 0.$$

Thus the determinant factor cannot vanish off the real axis. Hence every zero parameter satisfies $\eta = 0$, i.e. $t \in \mathbb{R}$. Since $\Xi(t) = \xi(1/2 + it)$, this is equivalent to the critical-line location of the nontrivial zeros of ζ. □

Corollary 9.7 (Reduced arithmetic targets)**.** *Within the present framework, the direct Hilbert–Pólya route is completed by two independent arithmetic statements:*

(a) a faithful FANG construction for $W_{\mathbb{P}}$, equivalently a faithful prime-side square-norm factorisation;
(b) a CRE determinant zero-set identity for Ξ.

The nilpotent shifted-root route replaces the first target by the more structured pair:

(a') a faithful shifted-root datum K' with $K'^2 = K_{\mathbb{P}} - I$;
(b') faithful nilpotent confinement of the self-adjoint defect

$$\Delta_{K'} = K_{\mathbb{P}} - (K' + J)(K' + J)^*.$$

Under these two shifted-root inputs, FANG follows from Theorem 7.4. NGC shows that diagonal compensation may be used without changing the genuine prime kernel, but it does not by itself remove the need for a faithful prime-side positivity mechanism.

Corollary 9.8 (Nilpotent-defect Hilbert–Pólya closure)**.** *Assume:*

(i) the prime-side kernel representation of Hypothesis 4.1 holds;
(ii) there exists a faithful shifted-root datum K' with $K'^2 = K_{\mathbb{P}} - I$;

(iii) the shifted-root factorisation defect

$$\Delta_{K'} = K_{\mathbb{P}} - (K' + J)(K' + J)^*$$

is faithfully confined to a nilpotent defect ideal;
(iv) the normal shifted-root spectral model applies;
(v) the induced translation representation is strongly continuous;
(vi) the CRE determinant zero-set identity of Hypothesis 9.5 holds.

Then every zero of $\Xi(t)$ is real. Equivalently, the Riemann Hypothesis follows under these hypotheses.

Proof. By Corollary 7.7, the nilpotent defect confinement gives

$$K_{\mathbb{P}} = AA^*,$$

and hence FANG holds with

$$\mathcal{A}_{\mathbb{P}} = A^*.$$

The GNS construction therefore gives a Hilbert space. By strong continuity, Theorem 8.4 gives a self-adjoint Hilbert–Pólya generator H_{HP}. The determinant identity and Lemma 9.4 then imply, exactly as in Theorem 9.6, that $\Xi(t)$ has no zero with nonzero imaginary part. Hence all zero parameters of Ξ are real. □

10. Non-circularity requirements and limitations

The framework above is intentionally conditional. The following requirements mark the boundary between a faithful arithmetic construction and a circular or regularised one.

(NC1) The kernel $K_{\mathbb{P}}$, the gauge datum D, and the transform $\mathcal{A}_{\mathbb{P}}$ must be constructed from prime-side data, the archimedean gamma factor, and functional-equation symmetry, not from the zero ordinates.
(NC2) The determinant identity must be obtained by trace-formula or explicit-formula comparison, not by imposing the Riemann zeros as the spectrum of H_{HP}.
(NC3) NGC must preserve $K_{\mathbb{P}}$ exactly. Replacing $K_{\mathbb{P}}$ by $K_{\mathbb{P}} + \lambda I$ proves only a regularised statement and is not a faithful prime-channel positivity proof.
(NC4) The compensated block must recover $K_{\mathbb{P}}$ as a genuine positive channel, not as a signed difference of positive blocks.
(NC5) In the infinite-dimensional setting, all gauge, dominance, square-root, and determinant statements must specify domains, closures, form bounds, and continuity hypotheses.
(NC6) The nilpotent defect ideal $\mathfrak{N}$ must be specified before the closure argument and constructed from the prime-side channel algebra, the null-pair gauge structure, the shifted-root datum, and functional-equation symmetry. It may not be defined post hoc by imposing $\Delta_{K'} = 0$, nor by using the zero ordinates of Ξ.

Component	Status here	Remaining issue
RH as real-zero property of Ξ	recalled	none
Local slice and CRE calculus	used locally	global AXIOM details are external to this paper
NGC identity $D + J(JD) = 0$	proved	none in the bounded setting
Kernel preservation under NGC	proved	common domain control in unbounded settings
Compensated block diagonalisation	proved	form-domain version must be stated carefully
Auxiliary-channel regularisation	proved under lower-boundedness	lower-bound or form-bound verification
Gauge-created prime positivity	not claimed	positivity of $K_{\mathbb{P}}$ remains arithmetic
Compensated dominance $\Rightarrow K_{\mathbb{P}} \succeq 0$	proved	dominance must be verified arithmetically
Shifted-root defect criterion	proved algebraically	existence and faithfulness of K' remain separate
Skew-channel reduction $K_{\mathbb{P}} - AA^* = 2AS$	proved algebraically	arithmetic control of S remains separate
Pure-skew projection classification	proved algebraically	projection branch is null, not negative; nilpotent confinement reduces the remaining issue to construction of the faithful defect ideal
Remaining fractional spectral obstruction	closed under nilpotent confinement	verify faithful nilpotent confinement of $\Delta_{K'}$
Self-adjoint nilpotent collapse	proved	bounded/form-domain version must be controlled in infinite dimension
Nilpotent defect confinement $\Delta_{K'} \in \mathfrak{N}$, $\mathfrak{N}^q = \{0\}$	stated as faithful arithmetic target	construct $\mathfrak{N}$ non-circularly from prime-side data
FANG from nilpotent confinement	proved	depends on faithful shifted root and faithful defect ideal
Pure-skew polynomial defect $\Delta_{K'} = -2B(B + I)$	proved	requires pure-skew reducing branch and J-commutation

Nilpotent exclusion of fractional branch	proved in normal spectral model	verify normal shifted-root spectral decomposition arithmetically
Direct FANG square-norm factorisation	conditional target	construct $\mathcal{A}_{\mathbb{P}}$ non-circularly, unless using the nilpotent shifted-root route
GNS self-adjoint generator	conditional on FANG and strong continuity	verify admissibility and completion details
CRE determinant zero-set identity	stated as hypothesis	trace/explicit-formula proof needed
Conditional RH closure	proved from hypotheses	depends on FANG and determinant identity
Nilpotent-defect Hilbert–Pólya closure	proved from hypotheses	depends on faithful shifted root, faithful defect ideal, and determinant identity

11. Conclusion

The NGC mechanism separates diagonal compensation from genuine positivity. The null-pair identity

$$D + J(JD) = 0$$

allows one to move mass between the Hermitian and skew-Hermitian channels while preserving the combined kernel

$$K_{\mathbb{P}} = S_K + JS_{KK}.$$

The compensated block has spectral channels

$$K_{\mathbb{P}} \qquad \text{and} \qquad S_K - JS_{KK} + 2D.$$

Hence the gauge can regularise the auxiliary channel under suitable lower-boundedness assumptions, but it cannot alter or fake the positivity of the faithful prime channel.

The nilpotent refinement closes the shifted-root route conditionally. The factorisation defect

$$\Delta_{K'} := K_{\mathbb{P}} - AA^*$$

is self-adjoint. Therefore, if it is confined to a faithful nilpotent defect ideal

$$\mathfrak{N}^q = \{0\},$$

then

$$\Delta_{K'}^q = 0 \qquad \Longrightarrow \qquad \Delta_{K'} = 0.$$

Consequently,

$$K_{\mathbb{P}} = AA^*.$$

Equivalently, with

$$\mathcal{A}_{\mathbb{P}} = A^*,$$

one obtains the faithful prime alignment factorisation

$$W_{\mathbb{P}}(f^\sharp * f) = \|\mathcal{A}_{\mathbb{P}} f\|^2.$$

Thus FANG follows from faithful shifted-root construction together with faithful nilpotent defect confinement.

This changes the logical role of positivity. In the original formulation, FANG appeared as an independent arithmetic hypothesis. In the nilpotent-refined formulation, FANG becomes the consequence of vanishing shifted-root defect. The arithmetic burden is therefore transferred from direct positivity to the construction of a faithful shifted-root datum and a faithful nilpotent defect ideal.

On pure-skew branches the same collapse takes the polynomial form

$$\Delta_{K'} = -2B(B + I).$$

Nilpotent confinement therefore forces

$$B(B + I) = 0,$$

hence

$$B = -P, \qquad P = P^* = P^2.$$

Therefore

$$K_{\mathbb{P}} = I - P = AA^* \succeq 0.$$

In the normal shifted-root spectral model, the residual fractional branch is excluded:

$$\mathrm{Spec}(K_{\mathbb{P}}) \cap \big((-\infty, 0) \cup (0, 1)\big) = \varnothing.$$

Thus the previously remaining shifted-root obstruction is closed, conditional on faithful nilpotent confinement.

Under the resulting FANG factorisation, the GNS construction gives a Hilbert space and, under strong continuity, Stone's theorem gives a self-adjoint Hilbert–Pólya generator. Under an additional CRE determinant zero-set identity for Ξ, off-axis coercivity of the HP–CRE pencil excludes non-real zero parameters.

Therefore the logical structure of the framework becomes

$$\text{NGC channel decomposition} + \text{faithful shifted root} + \text{nilpotent defect confinement}$$
$$\Longrightarrow \text{FANG positivity}.$$

Together with determinant realisation,

$$\text{FANG positivity} + \text{CRE determinant zero-set identity}$$
$$\Longrightarrow \text{Hilbert–Pólya closure}.$$

Equivalently,

$$\text{NGC} + \text{faithful shifted root} + \text{nilpotent defect confinement} + \text{determinant realisation} \Longrightarrow \text{conditional RH}.$$

The remaining non-circular arithmetic tasks are the construction of a faithful shifted root, the construction of a faithful nilpotent defect ideal, and the proof of the CRE determinant zero-set identity. In particular, the residual fractional obstruction is no longer an independent target; under the nilpotent refinement it is absorbed into the faithful defect-confinement problem.

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