CRE–SPD Reduction and Cholesky Inverse Reconstruction for Cayley–Dickson Tensor Operators

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Abstract

We develop a Conjugation–Real Embedding (CRE) framework for structured SPD reduction and inverse reconstruction of Cayley–Dickson tensor operators. The construction transports admissible complex, quaternionic, octonionic, sedenionic, and higher Cayley–Dickson ALH-slice operators into faithful real block representations without changing the underlying algebraic structure. The central result is an equivalence between Hermitian positive definiteness on an ALH slice and symmetric positive definiteness of the associated CRE representation.

This equivalence allows inverse reconstruction through positive-definite kernels, Cholesky factorisation, Schur-complement decompositions, and null-gauge-regularised SPD channels. To control failure of positivity and invertibility, we introduce Null-Pair Gauge Compensation (NGC) and projection-valued defect operators. We prove that defect-freeness, strict SPD positivity, invertibility, and Cholesky reconstruction are equivalent manifestations of the same structural condition.

Although the displayed CRE matrix has doubled real dimension, it stores only symmetric and skew channels rather than four independent real blocks. The resulting storage-neutral representation gives a precise setting for comparing structured inverse computation with generic doubled real inversion. The framework preserves cubic worst-case dense complexity, but identifies leading-constant gains from Cholesky factorisation and practical advantages from pivot-free SPD computation, without claiming a subcubic dense inverse algorithm.

1 Introduction

The purpose of this work is to develop the Conjugation–Real Embedding (CRE) framework as a structured inverse-recovery theory for tensor and operator systems. The central idea is that algebraic operators may be transported into faithful real block representations whose positivity, invertibility, and reconstruction properties can be analysed through symmetric positive-definite kernels.

The primary objective is computational rather than merely representational. By reducing admissible operator problems to SPD kernels, CRE allows inverse reconstruction through Cholesky factorisation, Schur-complement decompositions, and structured positive-definite linear algebra. The resulting framework applies not only to ordinary complex operators but also to quaternionic, octonionic, sedenionic, and higher Cayley–Dickson tensor operators represented through ALH local slices.

The framework is built on three preceding structural layers. First, the AXIOM programme isolates index-ordered multiplication, local Hurwitz slices, and symmetric block real embeddings as a common computational model for complex, Clifford, Cayley–Dickson, and tensor-valued operators [5]. Second, signed-XOR structural constants show that many Clifford and Cayley–Dickson basis

products share a common elementary two-group support law, with all algebra-specific information carried by a sign twist [8]. Third, ALH quarter-turn atlases show that admissible support-toggle directions generate skew-adjoint complex structures on homogeneous two-plane decompositions [6].

The present paper extracts the CRE part as a self-contained operator theory. Its role is to turn algebraic and signed-XOR data into faithful real block operators whose positivity, defect structure, and invertibility can be analysed through symmetric positive-definite kernels. The construction is not limited to ordinary complex operators: quaternionic, octonionic, sedenionic, and higher Cayley–Dickson local slices are represented faithfully once the ALH slice or support-toggle quarter-turn chart is fixed [6]. In this formulation, ALH isolates the local slice structure on which CRE acts, so the representation theory is developed at the slice/operator level rather than through global algebraic properties. The Hilbert–Pólya application and the prime-side positivity problem are not treated as the foundation here; rather, they are downstream applications of the CRE–NGC–defect mechanism [7].

The main contributions of the paper are:

1. intrinsic $*$-isomorphic operator embedding;
2. null-pair gauge compensation;
3. projection-valued defect theory;
4. SPD reduction;
5. inverse reconstruction;
6. Cayley–Dickson tensor inverse recovery through ALH-slice CRE;
7. storage-neutral symmetric–skew channel representation;
8. Schur-complement and block-inverse Big-O/leading-constant complexity consequences.

2 Conjugation–Real Embedding

Let

$$A = B + iC$$

be a bounded complex operator. This is the standard block form underlying realification and matrix-function treatments [4, 3].

Definition 2.1 (CRE)**.** *The Conjugation–Real Embedding of A is*

$$\mathrm{CRE}(A) = \begin{pmatrix} B & -C \\ C & B \end{pmatrix}.$$

Theorem 2.2 (Faithful Embedding)**.** *The map*

$$\mathrm{CRE} : \mathcal{B}(H) \to \mathcal{B}(H_{\mathbb{R}} \oplus H_{\mathbb{R}})$$

is an injective real $*$-homomorphism.

Proof. Write

$$A = B + iC, \qquad D = E + iF.$$

A direct computation gives

$$\mathrm{CRE}(A + D) = \mathrm{CRE}(A) + \mathrm{CRE}(D)$$

and

$$\mathrm{CRE}(AD) = \mathrm{CRE}(A)\,\mathrm{CRE}(D).$$

Furthermore,

$$A^* = B^T - iC^T$$

implies

$$\mathrm{CRE}(A^*) = \mathrm{CRE}(A)^T.$$

If $\mathrm{CRE}(A) = 0$ then $B = C = 0$, hence $A = 0$. Therefore CRE is injective and preserves multiplication and adjoints. □

Theorem 2.3 (Spectrum Preservation)**.** *For every bounded operator A, the complex spectrum of the CRE lift is the complex spectrum of A under the canonical slice identification. Equivalently,*

$$\lambda \in \sigma(A) \quad \Longleftrightarrow \quad \mathrm{CRE}(\lambda I) \in \sigma_{\mathrm{slice}}(\mathrm{CRE}(A)).$$

In finite dimension, the real characteristic polynomial of $\mathrm{CRE}(A)$ *is*

$$\det_{\mathbb{R}}(tI - \mathrm{CRE}(A)) = \det_{\mathbb{C}}(tI - A)\overline{\det_{\mathbb{C}}(tI - A)}.$$

Proof. The eigenvalue equation

$$A(x + iy) = \lambda(x + iy)$$

is equivalent to the real block system defined by $\mathrm{CRE}(A)$. Thus invertibility of $A - \lambda I$ and invertibility of $\mathrm{CRE}(A) - \mathrm{CRE}(\lambda I)$ are equivalent. □

Theorem 2.4 (Invertibility Equivalence)**.** *An operator A is invertible if and only if* $\mathrm{CRE}(A)$ *is invertible. Moreover,*

$$\mathrm{CRE}(A^{-1}) = \mathrm{CRE}(A)^{-1}.$$

Proof. Since CRE is a faithful algebra homomorphism,

$$A^{-1}A = I$$

implies

$$\mathrm{CRE}(A^{-1})\,\mathrm{CRE}(A) = I.$$

The converse follows from injectivity. □

3 CRE-Admissible Operators

The later positivity and reconstruction results require a precise class of operators. The following definition records the finite-dimensional bounded setting used in this paper.

Definition 3.1 (CRE-Admissible Operator)**.** *An operator A is called CRE-admissible if it satisfies the following conditions:*

1. *A acts on a finite-dimensional real or slice-complex Hilbert space;*
2. *A admits a decomposition $A = B + iC$ inside a fixed local complex slice with $i^2 = -1$;*
3. *the CRE block $\mathrm{CRE}(A)$ is a well-defined real matrix;*
4. *adjoints and products are preserved by the CRE map on the algebra generated by A and A^*.*

Remark 3.2 (Bounded finite-dimensional convention)**.** *The present paper works in the finite-dimensional bounded setting. The corresponding unbounded theory requires common invariant domains, closed forms, and relative form bounds. Those analytic domain issues are deliberately separated from the algebraic CRE mechanism.*

4 Algebraic Properties of CRE

The CRE construction is not merely a realification procedure. It preserves the algebraic, spectral, and invertibility structure of the original operator algebra.

The preceding results imply that operator-theoretic questions may be transferred from complex or higher-algebra settings into structured real block systems without loss of information.

5 CRE as an Intrinsic $*$-Isomorphism

The CRE construction is often written in coordinates. The essential point, however, is intrinsic: once a local complex structure is fixed, CRE is the real form of a slice-linear operator algebra.

Definition 5.1 (Slice Complex Structure)**.** *Let E be a real Hilbert space. A slice complex structure on E is an orthogonal operator J satisfying*

$$J^2 = -I, \qquad J^T = -J.$$

The associated slice-linear operator algebra is

$$\mathcal{B}_J(E) = \{T \in \mathcal{B}(E) : TJ = JT\}.$$

Theorem 5.2 (Intrinsic CRE $*$-Isomorphism)**.** *Let E be a finite-dimensional real Hilbert space with slice complex structure J. Then the assignment*

$$B + iC \longmapsto \begin{pmatrix} B & -C \\ C & B \end{pmatrix}$$

identifies the complex operator algebra on the slice with the real block algebra commuting with the standard block complex structure

$$\mathbb{J} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}.$$

This identification is an injective real $*$-isomorphism onto its image.

Proof. The proof is internal to the slice. The equations $J^2 = -I$ and $J^T = -J$ make E into a complex Hilbert space by defining multiplication by i to be J. A slice-linear operator is exactly a real operator commuting with this complex structure. In a J-adapted real orthonormal basis, such an operator has the block form

$$\begin{pmatrix} B & -C \\ C & B \end{pmatrix}.$$

Multiplication, addition, and adjoints are computed inside the same block algebra. The block calculation in the Faithful Embedding theorem shows preservation of products and adjoints, while injectivity follows because the two real blocks B and C are recovered uniquely from the image. Hence the construction is a $*$-isomorphism onto the CRE image. □

Remark 5.3 (Representation Rather Than Construction)**.** *The significance of the $*$-isomorphism theorem is that CRE does not create a new algebraic structure. The underlying algebra already exists on the original slice-linear operator algebra. CRE merely transports that structure into a faithful real block representation.*

What CRE preserves is the algebra itself: multiplication, involution, identities, invertibility relations, and all consequences that follow from the transported algebraic structure. CRE does not create additional regularity, associativity, commutativity, alternativity, or any other special algebraic property. If such properties are present, they are inherited from the original algebra; if they are absent, CRE does not manufacture them.

Products, adjoints, spectra, positivity properties, and invertibility are preserved because the same algebra is represented in different coordinates.

Remark 5.4 (Intrinsic content)**.** *The CRE map is therefore not merely a coordinate convention. Coordinates display the map, but the object being represented is the slice-linear $*$-algebra determined by the quarter-turn structure $J^2 = -I$ and $J^T = -J$.*

Theorem 5.5 (Rotation-Chart and Null-Gauge Covariance)**.** *Let two CRE charts be determined by quarter-turn structures or null-gauge presentations satisfying the CRE relations. Then the resulting CRE images are related by an orthogonal signed block change of coordinates. In particular, both charts define real $*$-isomorphic CRE image algebras.*

However, inverse reconstruction is chart-dependent: to recover the original algebraic operator from a CRE inverse, one must decode using the same rotation direction and null-gauge mapping used in the forward embedding.

Proof. A change of quarter-turn direction or null-gauge presentation that preserves the CRE relations changes only the real block coordinates of the represented slice-linear algebra. Such a change is implemented by an orthogonal signed block transformation Q. Hence a CRE representative F is replaced by

$$F' = Q^T F Q.$$

For the adjoint,

$$(F')^T = Q^T F^T Q,$$

and multiplication is also preserved:

$$F'G' = (Q^T F Q)(Q^T G Q) = Q^T (FG) Q.$$

Thus the two CRE images are $*$-isomorphic.

For reconstruction, however, the inverse block must be interpreted in the same chart in which it was produced. If the forward embedding used chart Q, then

$$(F')^{-1} = Q^T F^{-1} Q.$$

Decoding F'^{-1} with a different chart would identify the block entries with different algebraic directions and therefore recover a conjugate representation rather than the original operator. Hence the $*$-isomorphism is chart-covariant, while reconstruction requires chart consistency. □

Remark 5.6 (Chart consistency)**.** *The direction of rotation and the null-gauge mapping are not additional algebraic content of the represented operator; they are coordinate choices inside the CRE representation. Any choice satisfying the CRE relations gives a valid $*$-isomorphic image. Nevertheless, numerical inversion is a round-trip operation, so the decoding step must use the same chart and gauge convention as the encoding step.*

6 Compatibility with Arithmetic Local Hurwitz Structures

The CRE framework is compatible with the Arithmetic Local Hurwitz (ALH) formalism developed for signed-XOR and support-toggle algebraic structures [6]. In ALH theory, admissible directions generate quarter-turn operators satisfying

$$R_\delta^2 = -I.$$

These operators play the role of local complex structures on support-toggle planes.

Theorem 6.1 (Local Complex Slice Principle)**.** *Let e be an ALH generator satisfying*

$$e^2 = -1, \qquad e^* = -e.$$

Then

$$\mathbb{R}1 \oplus \mathbb{R}e \cong \mathbb{C}$$

as a real $*$-algebra, and the corresponding quarter-turn operator is preserved by CRE.

This is the fundamental local-slice result established in the ALH theory [6].

Proof. The multiplication rule

$$(a + be)(c + de) = (ac - bd) + (ad + bc)e$$

is exactly the complex multiplication law under $a+be \mapsto a+bi$. The involution $e^* = -e$ corresponds to complex conjugation. CRE is a $*$-isomorphism on the associated local slice, so it preserves the induced quarter-turn structure. □

Theorem 6.2 (ALH Compatibility)**.** *Let*

$$R_\delta^2 = -I$$

be a right-ALH-admissible quarter-turn operator. Then

$$\mathrm{CRE}(R_\delta)^T \,\mathrm{CRE}(R_\delta) = I.$$

Consequently every ALH quarter-turn operator generates an SPD kernel equal to the identity.

The quarter-turn identities are inherited from the ALH atlas construction [6].

Proof. Since

$$R_\delta^2 = -I, \qquad R_\delta^\dagger = -R_\delta,$$

we obtain

$$R_\delta^\dagger R_\delta = I.$$

Because CRE preserves multiplication and adjoints,

$$\mathrm{CRE}(R_\delta)^T \,\mathrm{CRE}(R_\delta) = \mathrm{CRE}(I) = I.$$

The result follows. □

Theorem 6.3 (Support-Toggle Stability)**.** *Let*

$$J(e_\alpha) = \sigma_\alpha e_{\alpha+\delta}$$

be a support-toggle ALH quarter-turn operator. Then the associated CRE kernel is invariant under the support-toggle decomposition.

Proof. The support decomposition partitions the basis into two-dimensional ALH quarter-turn planes. On each plane the operator acts as a standard complex structure

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Hence the corresponding CRE kernel equals the identity on every orbit plane. Direct summation over all support orbits yields global invariance. □

The preceding theorem establishes that ALH quarter-turn geometry survives unchanged under CRE reduction. Consequently the progression

$$\text{Signed-XOR} \longrightarrow \text{ALH} \longrightarrow \text{CRE}$$

preserves the underlying support-level structure while converting algebraic questions into positive-definite operator questions.

7 CRE for Quaternionic and Cayley–Dickson ALH Slices

The intrinsic CRE mechanism is not restricted to the ordinary complex field. Quaternionic, octonionic, sedenionic, and higher Cayley–Dickson systems all contain ALH local slices and support-toggle quarter-turn charts. CRE applies to these local structures because ALH supplies the relevant two-dimensional $*$-algebraic complex structure [6].

Definition 7.1 (ALH-Faithful CRE Representation)**.** *Let A be a real $*$-algebraic system and let*

$$C_e = \mathbb{R}1 \oplus \mathbb{R}e$$

be an ALH slice satisfying

$$e^2 = -1, \qquad e^* = -e.$$

A CRE representation is ALH-faithful on C_e if it is injective and satisfies

$$\mathrm{CRE}(xy) = \mathrm{CRE}(x)\,\mathrm{CRE}(y), \qquad \mathrm{CRE}(x^*) = \mathrm{CRE}(x)^T$$

for all $x, y \in C_e$.

Theorem 7.2 (ALH Slice $*$-Isomorphism)**.** *Let e satisfy*

$$e^2 = -1, \qquad e^* = -e.$$

Then

$$C_e = \mathbb{R}1 \oplus \mathbb{R}e$$

is $*$-isomorphic to $\mathbb{C}$, and

$$C_e \cong_* \mathrm{CRE}(C_e).$$

This theorem is the CRE realisation of the local-slice principle established in the ALH framework [6].

Proof. For $x = a + be$ and $y = c + de$, the ALH relations give the usual complex multiplication law

$$xy = (ac - bd) + (ad + bc)e.$$

The involution is $x^* = a - be$. Therefore $a + be \mapsto a + bi$ identifies C_e with $\mathbb{C}$. The existence of such local slices is supplied by the ALH theory [6]. The Intrinsic CRE $*$-Isomorphism theorem then yields

$$C_e \cong_* \mathrm{CRE}(C_e).$$

□

Theorem 7.3 (Quaternionic, Octonionic, and Sedenionic Compatibility)**.** *Every quaternionic, octonionic, sedenionic, or higher Cayley–Dickson ALH generator determines a local CRE $*$-isomorphism on its ALH slice.*

In particular,

$$\mathbb{R}1 \oplus \mathbb{R}e \cong_* \mathrm{CRE}(\mathbb{R}1 \oplus \mathbb{R}e).$$

Likewise every ALH support-toggle quarter-turn operator J satisfies

$$J^2 = -I, \qquad J^T = -J, \qquad \mathrm{CRE}(J)^T \,\mathrm{CRE}(J) = I.$$

The result follows from applying the ALH local-slice theorem uniformly across the Cayley–Dickson hierarchy [6].

Proof. The ALH theory guarantees the existence of local slices satisfying $e^2 = -1$ and $e^* = -e$ throughout the quaternionic, octonionic, sedenionic, and higher Cayley–Dickson hierarchy [6]. Applying the preceding theorem on each slice yields the local $*$-isomorphism. The quarter-turn identities follow from ALH compatibility and adjoint preservation under CRE. □

Remark 7.4 (Role of ALH)**.** *CRE preserves the algebra that is presented to it. The purpose of ALH is to isolate local complex slices on which the relevant algebraic structure is expressed. Consequently the CRE statement is local and slice-theoretic: it preserves the algebraic structure of the ALH slice without asserting any new global algebraic properties. In particular, CRE depends on ALH only through the extracted local slice structure; CRE preserves the algebra presented by the slice, rather than introducing any new algebraic properties.*

8 HPD–SPD Equivalence under CRE

Definition 8.1 (HPD Slice Operator)**.** *A CRE-admissible operator A is Hermitian positive definite, abbreviated HPD, if*

$$A^* = A$$

and

$$\langle Ax, x\rangle > 0$$

for every nonzero vector x in the slice Hilbert space.

Theorem 8.2 (HPD–SPD Equivalence)**.** *Let A be a finite-dimensional CRE-admissible slice operator. Then*

$$A \text{ is HPD} \quad \Longleftrightarrow \quad \mathrm{CRE}(A) \text{ is real SPD.}$$

Moreover, for every slice vector $z = x + iy$ and its real coordinate vector $\widehat{z} = (x, y)$,

$$\widehat{z}^T \,\mathrm{CRE}(A)\widehat{z} = \mathrm{Re}\langle Az, z\rangle.$$

Proof. Write $A = B + iC$. If $A^* = A$, then $B^T = B$ and $C^T = -C$. Hence

$$\mathrm{CRE}(A) = \begin{pmatrix} B & -C \\ C & B \end{pmatrix}$$

is real symmetric. The real-coordinate identity

$$\widehat{z}^T \,\mathrm{CRE}(A)\widehat{z} = \mathrm{Re}\langle Az, z\rangle$$

shows that HPD of A implies SPD of $\mathrm{CRE}(A)$.

Conversely, if $\mathrm{CRE}(A)$ is real SPD, then it is symmetric. By adjoint preservation,

$$\mathrm{CRE}(A^*) = \mathrm{CRE}(A)^T = \mathrm{CRE}(A).$$

Injectivity gives $A^* = A$. The quadratic-form identity then gives positivity of $\langle Az, z\rangle$ for every nonzero slice vector z. Hence A is HPD. □

Remark 8.3 (HPD–SPD as a Consequence of $*$-Isomorphism)**.** *The HPD–SPD equivalence is not an additional structure imposed by CRE. It is a consequence of the intrinsic $*$-isomorphism. Since CRE identifies the original slice algebra with its real block image, the Hermitian cone of the original algebra is carried exactly to the symmetric cone of the real image, and strict positivity is preserved by the transported quadratic form.*

Theorem 8.4 (ALH-Hermitian–SPD Equivalence)**.** *Let A be a finite-dimensional ALH-admissible Hermitian operator on a local ALH slice or support-toggle quarter-turn chart. Then*

$$A \text{ is Hermitian positive definite on the ALH slice} \quad \Longleftrightarrow \quad \mathrm{CRE}(A) \text{ is real SPD.}$$

Consequently the HPD–SPD equivalence extends uniformly to quaternionic, octonionic, sedenionic, and higher Cayley–Dickson ALH slices.

Proof. By the ALH Slice $*$-Isomorphism theorem, every ALH slice has the local form

$$C_e = \mathbb{R}1 \oplus \mathbb{R}e \cong_* \mathbb{C}.$$

The intrinsic CRE $*$-isomorphism identifies this slice with its faithful real block image. Hermitian elements are carried to symmetric real blocks, and positive quadratic forms are carried to positive real quadratic forms. Therefore Hermitian positive definiteness on the ALH slice is equivalent to real SPD positivity of the CRE image.

Quaternionic, octonionic, sedenionic, and higher Cayley–Dickson cases follow by applying the same ALH slice argument to the corresponding local generator or support-toggle quarter-turn chart [6]. □

Corollary 8.5 (Cayley–Dickson Hermitian Reduction)**.** *Hermitian positive-definite operators over quaternionic, octonionic, sedenionic, and higher Cayley–Dickson ALH slices may be reduced by CRE to real SPD operators. Hence every SPD theorem used in this paper applies to those Hermitian ALH-slice operators after CRE.*

Proof. This is the ALH-Hermitian–SPD Equivalence theorem applied to Cayley–Dickson ALH slices. □

Corollary 8.6 (CRE Reduction of Positive Linear Algebra)**.** *Let A be a finite-dimensional CRE-admissible Hermitian positive-definite (HPD) operator. Then* $\mathrm{CRE}(A)$ *is symmetric positive-definite (SPD), and every SPD result for* $\mathrm{CRE}(A)$ *has an HPD analogue for A.*

Proof. By the HPD–SPD Equivalence theorem, A is HPD if and only if $\mathrm{CRE}(A)$ is real SPD. Since CRE is an intrinsic $*$-isomorphism onto its image, any theorem depending only on the real SPD structure of $\mathrm{CRE}(A)$ transfers back to the HPD slice operator A through the inverse CRE identification. □

Theorem 8.7 (Spectral Transfer Principle)**.** *Let A be a finite-dimensional CRE-admissible HPD operator. Then:*

1. *A and* $\mathrm{CRE}(A)$ *have the same positive spectrum under the canonical slice identification;*
2. *A is HPD if and only if* $\mathrm{CRE}(A)$ *is SPD;*
3. *every spectral statement for SPD matrices transfers to HPD slice operators through CRE, and conversely;*
4. *functional calculus is preserved:*

$$\mathrm{CRE}(f(A)) = f(\mathrm{CRE}(A))$$

for every polynomial and, by spectral approximation, every continuous functional-calculus expression.

Proof. Spectrum preservation was established earlier. The HPD–SPD equivalence theorem identifies the positive cones on both sides. Since CRE is an injective $*$-isomorphism onto its image, algebraic identities, adjoints, inverses, powers, and polynomial functional calculus are preserved. Spectral approximation then extends the result to continuous functional calculus in finite dimension. □

Corollary 8.8 (Transfer of SPD Theory)**.** *Let A be HPD and let $M = \mathrm{CRE}(A)$. Then every theorem depending only on the SPD structure of M admits an equivalent HPD formulation for A.*

In particular, the following notions are preserved under CRE:

1. *Cholesky factorization;*
2. *spectral decomposition;*
3. *positive square roots;*
4. *matrix logarithms and exponentials;*
5. *Schur complements;*
6. *condition numbers;*
7. *SPD stability estimates;*
8. *positivity-preserving functional calculus.*

Conversely, every HPD statement may be translated into an SPD statement through CRE.

Proof. The result follows from the combination of spectrum preservation, adjoint preservation, the intrinsic $*$-isomorphism theorem, and the HPD–SPD equivalence theorem. □

Remark 8.9 (CRE Philosophy)**.** *The practical meaning of CRE is that HPD operator theory and SPD matrix theory become equivalent languages. One may therefore solve a problem in whichever category is more convenient and transfer the result through CRE without loss of spectral, positivity, or invertibility information.*

This equivalence is possible precisely because CRE is a faithful $*$-isomorphic representation mechanism rather than a construction of a new algebra.

9 Null-Pair Gauge Compensation

Null-Pair Gauge Compensation (NGC) records the fact that certain channel decompositions may change auxiliary blocks while leaving the genuine CRE kernel unchanged.

Theorem 9.1 (NGC Kernel Preservation)**.** *Let*

$$K = S_K + JS_{KK}$$

be a slice operator, where $S_K = S_K^*$, $S_{KK}^* = -S_{KK}$, and all terms commute with the local complex structure J. Let $D = D^*$ commute with J, and define

$$S_{K,D} = S_K + D, \qquad S_{KK,D} = S_{KK} + JD.$$

Then

$$S_{K,D} + JS_{KK,D} = K.$$

Thus NGC changes the channel decomposition but not the genuine kernel.

Proof. Using $J^2 = -I$ and $JD = DJ$, one obtains

$$S_{K,D} + JS_{KK,D} = S_K + D + J(S_{KK} + JD) = S_K + D + JS_{KK} + J^2D = S_K + JS_{KK}.$$

□

Theorem 9.2 (Compensated Channel Decomposition)**.** *Set*

$$A_D = S_K + D, \qquad R_D = JS_{KK} - D.$$

Then the compensated block

$$C_D(K) = \begin{pmatrix} A_D & JR_D \\ -JR_D & A_D \end{pmatrix}$$

is orthogonally equivalent to

$$K \oplus (S_K - JS_{KK} + 2D).$$

Consequently NGC may regularize the auxiliary channel, but it cannot manufacture positivity or invertibility of the genuine channel K.

Proof. Use the orthogonal block change of variables

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} I & I \\ -J & J \end{pmatrix}.$$

A direct multiplication gives

$$U^T C_D(K) U = \begin{pmatrix} A_D + R_D & 0 \\ 0 & A_D - R_D \end{pmatrix}.$$

Since

$$A_D + R_D = S_K + JS_{KK} = K$$

and

$$A_D - R_D = S_K - JS_{KK} + 2D,$$

the decomposition follows. □

Definition 9.3 (Gauge Pair)**.** *A pair of operator components $(G, -G)$ is called a null-pair gauge contribution if its total observable contribution vanishes after CRE reduction.*

Definition 9.4 (Null-Pair Gauge Compensation)**.** *The elimination of mutually cancelling gauge pairs is called Null-Pair Gauge Compensation (NGC).*

Theorem 9.5 (Gauge Cancellation)**.** *Let*

$$T = T_0 + G - G$$

be a decomposition containing a null-pair gauge contribution. Then

$$\mathrm{CRE}(T) = \mathrm{CRE}(T_0).$$

Hence null-pair gauge components do not affect the reduced operator.

Proof. By linearity of CRE,

$$\mathrm{CRE}(T) = \mathrm{CRE}(T_0) + \mathrm{CRE}(G) - \mathrm{CRE}(G).$$

The last two terms cancel identically. □

10 Projection-Valued Defects

Definition 10.1 (Self-Adjointness Defect)**.** *For a candidate shifted root K' define its self-adjointness defect by*

$$\Delta(K') = K' - K'^*.$$

The defect vanishes precisely when K' is self-adjoint.

Theorem 10.2 (Shifted-Root Defect Identity)**.** *Let K' be a slice-linear operator commuting with J and suppose*

$$K'^2 = K - I.$$

Set

$$A = K' + J.$$

Then

$$K - AA^* = A(K' - K'^*).$$

Thus the factorization $K = AA^*$ holds precisely when the shifted-root self-adjointness defect is annihilated by A.

Proof. Since K' commutes with J and $J^2 = -I$,

$$(K' + J)(K' - J) = K'^2 + I = K.$$

Also

$$A^* = K'^* - J.$$

Hence

$$K' - J = A^* + (K' - K'^*).$$

Multiplying by $A = K' + J$ gives

$$K = A(K' - J) = AA^* + A(K' - K'^*),$$

which proves the identity. □

Corollary 10.3 (Skew-Channel Defect)**.** *Write*

$$K' = H + S, \qquad H = H^*, \qquad S^* = -S.$$

Then

$$K - AA^* = 2AS.$$

Thus the skew-Hermitian component is the full shifted-root obstruction.

Proof. Since $K' - K'^* = 2S$, the result follows from the shifted-root defect identity. □

Definition 10.4 (Projection-Valued Defect)**.** *A projection operator*

$$P = P^* = P^2$$

is called a projection-valued defect if it represents the obstruction preventing strict positivity of a CRE-reduced operator.

Theorem 10.5 (Positivity Obstruction)**.** *Let K be a self-adjoint CRE kernel. Then strict positivity fails if and only if there exists a nontrivial projection-valued defect*

$$P = P^* = P^2$$

contained in the null space of K.

Proof. If such a projection exists, every vector in its range belongs to the kernel of K, preventing strict positivity. Conversely, failure of strict positivity produces a nontrivial spectral projection associated with the zero eigenspace. □

Theorem 10.6 (Pure-Skew Projection Branch)**.** *Assume the shifted root is pure-skew, so $K' = S$, and write*

$$S = JB, \qquad B = B^*.$$

Then the shifted-root defect equation reduces to

$$B(B + I) = 0.$$

Consequently

$$B = -P$$

for an orthogonal projection $P = P^* = P^2$, and

$$K = I - P \succeq 0.$$

Proof. On the pure-skew branch, the shifted-root relation and the defect equation force the algebraic constraint $(S + J)S = 0$. Substituting $S = JB$ and using $J^2 = -I$ gives

$$(JB + J)JB = J(B + I)JB = 0.$$

Cancelling the invertible factor J and using commutation inside the slice yields

$$B(B + I) = 0.$$

Since B is self-adjoint, the spectral theorem implies that its spectrum is contained in $\{0, -1\}$; hence $B = -P$ for an orthogonal projection P. The shifted kernel is then $K = I - P$, which is positive semidefinite. □

11 SPD Reduction and Invertibility Reconstruction

Definition 11.1 (CRE Kernel)**.** *For an operator A, define the associated CRE kernel by*

$$\mathcal{K}(A) = \mathrm{CRE}(A)^T \, \mathrm{CRE}(A).$$

Theorem 11.2 (CRE–SPD Reduction)**.** *Let A be a CRE-admissible operator. Then*

$$\mathcal{K}(A) = \mathrm{CRE}(A)^T \, \mathrm{CRE}(A)$$

is symmetric and positive semidefinite.

If no nontrivial projection-valued defect is present, then

$$\mathcal{K}(A) \succ 0.$$

Hence the reduced kernel is symmetric positive definite.

Proof. Symmetry follows from

$$\mathcal{K}(A)^T = \big(\mathrm{CRE}(A)^T\,\mathrm{CRE}(A)\big)^T = \mathcal{K}(A).$$

For every vector x,

$$x^T\mathcal{K}(A)x = \|\,\mathrm{CRE}(A)x\|^2 \geq 0.$$

Therefore $\mathcal{K}(A)$ is positive semidefinite. If the null space contains no nontrivial projection-valued defect, the quadratic form vanishes only for $x = 0$, yielding strict positivity. □

Theorem 11.3 (Cholesky Reconstruction)**.** *Suppose*

$$\mathcal{K}(A) \succ 0.$$

Then there exists a unique Cholesky factorization

$$\mathcal{K}(A) = LL^T$$

with L lower triangular and positive diagonal.

Proof. This is the classical Cholesky theorem for symmetric positive-definite matrices; see, for instance, [1, 9, 2]. □

Theorem 11.4 (Invertibility Recovery)**.** *Let A be a CRE-admissible operator. If*

$$\mathcal{K}(A) \succ 0,$$

then A is invertible.

Moreover,

$$\mathrm{CRE}(A)^{-1} = \mathcal{K}(A)^{-1}\,\mathrm{CRE}(A)^T.$$

Consequently the inverse of A can be reconstructed from the SPD kernel.

Proof. Strict positivity implies

$$\ker \mathrm{CRE}(A) = \{0\}.$$

Hence $\mathrm{CRE}(A)$ is invertible. Using the identity

$$\mathcal{K}(A) = \mathrm{CRE}(A)^T\,\mathrm{CRE}(A),$$

one obtains

$$\mathcal{K}(A)^{-1}\,\mathrm{CRE}(A)^T\,\mathrm{CRE}(A) = I.$$

Thus

$$\mathrm{CRE}(A)^{-1} = \mathcal{K}(A)^{-1}\,\mathrm{CRE}(A)^T.$$

The invertibility equivalence theorem then implies invertibility of A itself. □

Theorem 11.5 (Cayley–Dickson Tensor Inverse Recovery)**.** *Let A be a finite-dimensional tensor operator over a quaternionic, octonionic, sedenionic, or higher Cayley–Dickson ALH slice. Suppose A is CRE-admissible and has no projection-valued defect. Then*

$$\mathcal{K}(A) = \mathrm{CRE}(A)^T\,\mathrm{CRE}(A)$$

is SPD, and the inverse of A is reconstructed from

$$\mathrm{CRE}(A)^{-1} = \mathcal{K}(A)^{-1}\,\mathrm{CRE}(A)^T.$$

Thus the CRE inverse-recovery mechanism applies uniformly to Cayley–Dickson tensor operators presented through ALH slices, provided reconstruction is performed in the same ALH slice chart and null-gauge convention used for the forward CRE embedding.

Proof. The ALH framework supplies the local slice or support-toggle chart on which the Cayley–Dickson tensor operator is represented. CRE preserves the slice algebra and sends the tensor operator to a faithful real block representative. By the CRE–SPD Reduction theorem, absence of a projection-valued defect implies that $\mathcal{K}(A)$ is SPD. The Invertibility Recovery theorem then gives the displayed inverse formula. Since CRE is faithful, decoding the CRE inverse in the same ALH chart recovers the inverse of the original tensor operator. □

11.1 Null-Gauge SPD Regularisation for Indefinite Symmetric Cores

A symmetric CRE core need not be positive definite. Consequently ordinary Cholesky factorisation may fail even when the underlying tensor operator is genuinely invertible. Null-pair gauge compensation removes this computational obstruction without changing the genuine kernel.

Definition 11.6 (Invertible Tensor Core)**.** *A CRE-admissible operator A is said to have an invertible tensor core if*

$$\ker \mathrm{CRE}(A) = \{0\}.$$

Equivalently, A is invertible by the Invertibility Equivalence theorem.

Theorem 11.7 (Null-Gauge SPD Regularisation)**.** *Let*

$$K = S_K + JS_{KK}, \qquad S_K = S_K^*, \qquad S_{KK}^* = -S_{KK},$$

with all terms commuting with the local complex structure J.

Assume that the corresponding tensor core is invertible. Then there exists a Hermitian gauge

$$D = D^*, \qquad DJ = JD,$$

such that the compensated auxiliary channel

$$S_K - JS_{KK} + 2D$$

is symmetric positive definite.

Moreover,

$$C_D(K) = \begin{pmatrix} A_D & JR_D \\ -JR_D & A_D \end{pmatrix}, \qquad A_D = S_K + D, \qquad R_D = JS_{KK} - D,$$

is invertible if and only if the genuine channel K is invertible.

Proof. By the Compensated Channel Decomposition theorem,

$$C_D(K) \sim_{\mathrm{orth}} K \oplus (S_K - JS_{KK} + 2D).$$

Since the setting is finite-dimensional, choose

$$D = \lambda I$$

with λ larger than the absolute value of the most negative eigenvalue of

$$S_K - JS_{KK}.$$

Then

$$S_K - JS_{KK} + 2D \succ 0.$$

Therefore the auxiliary channel becomes SPD.

Because orthogonal equivalence preserves invertibility, the compensated block is invertible exactly when the genuine channel K is invertible. The gauge modifies only the auxiliary channel and leaves the genuine kernel unchanged. □

Corollary 11.8 (Cholesky Recovery via NGC)**.** *Let A be a CRE-admissible operator with invertible tensor core.*

If the original symmetric CRE representation is indefinite and therefore not directly Cholesky-factorisable, then NGC produces an SPD auxiliary channel to which Cholesky factorisation applies.

Consequently inverse recovery may still be performed through the compensated SPD channel.

Proof. Apply the Null-Gauge SPD Regularisation theorem to obtain a compensated auxiliary channel

$$S_K - JS_{KK} + 2D \succ 0.$$

The Cholesky Reconstruction theorem applies to this SPD channel. Since the compensated block is orthogonally equivalent to

$$K \oplus (S_K - JS_{KK} + 2D),$$

and since the gauge leaves the genuine channel K unchanged, the inverse information is recovered through the compensated block decomposition and then restricted back to the genuine CRE channel. □

Remark 11.9 (What NGC Does Not Do)**.** *NGC does not create invertibility. If the original tensor operator is genuinely singular, no null gauge can manufacture an inverse. The role of NGC is narrower and precise: it removes a symmetric-indefinite computational obstruction and transfers the calculation into an SPD channel whenever a genuine inverse already exists.*

The preceding theorem together with the null-gauge SPD regularisation mechanism forms the central computational consequence of CRE. Rather than computing inverses directly in the original algebraic setting, one may construct the SPD kernel, perform Cholesky factorization, and recover the inverse through the associated CRE reconstruction formula. This converts invertibility analysis into positive-definite kernel analysis.

12 Storage and Block-Inverse Complexity

The CRE display

$$\mathrm{CRE}(A) = \begin{pmatrix} B & -C \\ C & B \end{pmatrix}$$

looks like a doubled real matrix. This does not mean that the representation stores twice as much algebraic information. The two off-diagonal blocks are determined by one skew channel, and the two diagonal blocks are determined by one symmetric channel. Thus CRE stores the pair (B, C), not four independent blocks.

Definition 12.1 (Symmetric–Skew Channel Storage)**.** *A CRE representation is called storage-neutral if its implementation stores only the independent channels*

$$B, \qquad C,$$

and reconstructs the block display

$$\begin{pmatrix} B & -C \\ C & B \end{pmatrix}$$

on demand.

Theorem 12.2 (Storage Neutrality)**.** *Let $A = B + iC$ be a CRE-admissible operator over a local complex, Clifford, or signed-XOR slice. Then the CRE block representation is information-equivalent to storing the original channel pair (B, C).*

In particular, although the displayed real block has twice the dimension, it contains no independent information beyond the original symmetric–skew channel data.

Proof. The CRE block is completely determined by the pair (B, C). Conversely, the pair (B, C) is recovered uniquely from the diagonal and off-diagonal blocks of $\mathrm{CRE}(A)$. The remaining two blocks are forced to be copies with signs. Therefore the CRE display is redundant as a full matrix but faithful as a structured representation. □

Remark 12.3 (Clifford and signed-XOR scope)**.** *The same storage-neutral interpretation applies to Clifford and signed-XOR sectors once a local ALH slice or support-toggle chart has been fixed. The support law and sign twist determine how the independent channels are assembled, while CRE records them as a structured real block representation.*

The block form also gives a standard route to inverse recovery. If

$$F = \begin{pmatrix} B & -C \\ C & B \end{pmatrix}$$

and B is invertible, then the Schur complement of B is

$$S = B + CB^{-1}C.$$

Whenever B and S are invertible, the inverse of F is given by the usual block-inverse formula

$$F^{-1} = \begin{pmatrix} B^{-1} - B^{-1}CS^{-1}CB^{-1} & B^{-1}CS^{-1} \\ -S^{-1}CB^{-1} & S^{-1} \end{pmatrix}.$$

Theorem 12.4 (Schur-Complement Inverse Path)**.** *Let $A = B+iC$ be CRE-admissible and suppose B and*

$$S = B + CB^{-1}C$$

are invertible. Then $\mathrm{CRE}(A)$ is invertible and its inverse is recovered by the Schur-complement formula above.

Proof. This is the standard block inverse formula applied to

$$F = \begin{pmatrix} B & -C \\ C & B \end{pmatrix}.$$

The Schur complement of the upper-left block B is

$$B - CB^{-1}(-C) = B + CB^{-1}C.$$

Substituting this Schur complement into the block inverse formula gives the displayed inverse. □

Corollary 12.5 (Complexity Consequence)**.** *In a storage-neutral CRE implementation, inverse computation may be organised through structured real SPD, Schur-complement, or compensated NGC-SPD kernels rather than through an unstructured doubled matrix.*

Thus the relevant complexity comparison is not between an $n \times n$ matrix and a fully independent $2n \times 2n$ real matrix, but between the original algebraic operator and its structured real channel representation. This permits leading-constant improvements whenever the real SPD or Schur-complement kernels exploit the CRE block constraints. Additional practical gains may arise from pivot-free SPD factorisation and the resulting reduction in memory-traffic and permutation overhead.

Proof. The storage-neutral representation stores only the independent channel pair (B, C). The inverse formulas above use the repeated-sign block structure rather than treating all four blocks as independent. Therefore the computational pathway can exploit real SPD or Schur-complement kernels while avoiding the memory and arithmetic cost of a generic dense $2n \times 2n$ matrix. □

12.1 Big-O and Leading-Constant Comparison

The asymptotic order of dense direct inversion is cubic. CRE does not change this order in the worst case. Its computational advantage is in the leading constant: once a problem is reduced to an SPD kernel, Cholesky factorisation replaces generic LU-type factorisation. For dense matrices, the leading operation count for Cholesky factorisation is approximately

$$\frac{1}{3}n^3,$$

whereas generic LU factorisation has leading count approximately

$$\frac{2}{3}n^3.$$

Thus the exponent remains cubic, but the principal dense-factorisation constant is reduced by roughly a factor of two whenever the CRE reduction legitimately reaches an SPD kernel. In addition, the structured real representative can use Schur-complement kernels instead of treating the displayed block as a generic doubled real matrix.

Let $A \in \mathbb{C}^{n\times n}$ be represented as

$$A = B + iC, \qquad B, C \in \mathbb{R}^{n\times n}.$$

A generic complex dense inverse costs

$$O(n^3)$$

complex arithmetic operations, which corresponds to a larger real-arithmetic leading constant. A naive unstructured realification would treat

$$\mathrm{CRE}(A) \in \mathbb{R}^{2n\times 2n}$$

as a generic dense real matrix whose leading arithmetic scale is proportional to

$$(2n)^3 = 8n^3.$$

The asymptotic order remains cubic, but the leading scale is eight times larger than the corresponding size-n dense real problem.

CRE is different: it stores only the independent channels (B, C) and exploits the repeated block pattern. Therefore its relevant cost is not the cost of a generic $2n \times 2n$ inverse, but the cost of structured real kernels acting on the independent channels.

Theorem 12.6 (Dense Direct Complexity Comparison)**.** *In the dense finite-dimensional setting, the following cost comparison holds:*

1. *generic complex inversion has cubic order $O(n^3)$ with complex-arithmetic constants;*
2. *naive unstructured realification remains cubic but has leading arithmetic scale proportional to $8n^3$ because the displayed matrix has dimension $2n$;*

3. *storage-neutral CRE inversion has cubic order $O(n^3)$ on the independent channel data; when reduced to an SPD kernel, the Cholesky leading constant is approximately $\frac{1}{3}n^3$, compared with approximately $\frac{2}{3}n^3$ for generic LU-type dense factorisation.*

Thus CRE does not improve the worst-case exponent below cubic for dense direct inversion, but it can reduce the real-arithmetic leading constant through Cholesky-type SPD factorisation and avoid the generic doubled-matrix cost.

Proof. Dense direct inversion and factorisation algorithms have cubic arithmetic order. Treating a $2n \times 2n$ real block as unstructured gives a nominal cubic cost proportional to $(2n)^3 = 8n^3$. In contrast, CRE stores and manipulates only the independent channel pair (B, C) and uses the forced block relations

$$\begin{pmatrix} B & -C \\ C & B \end{pmatrix}.$$

Consequently the computation is organised through structured channel operations, SPD kernels, Cholesky factors, or Schur complements rather than through a generic dense $2n \times 2n$ inverse. The exponent remains cubic, but, in the SPD case, Cholesky replaces a generic LU-type factorisation and changes the leading dense-factorisation constant from approximately $\frac{2}{3}n^3$ to approximately $\frac{1}{3}n^3$. □

Corollary 12.7 (SPD Kernel Cost)**.** *If the CRE problem is reduced to an SPD kernel*

$$\mathcal{K}(A) = \mathrm{CRE}(A)^T \,\mathrm{CRE}(A) \succ 0,$$

then Cholesky factorisation costs

$$\frac{1}{3}n^3 + O(n^2)$$

on the independent channel representation. By contrast, a generic LU-type dense factorisation has leading cost

$$\frac{2}{3}n^3 + O(n^2),$$

and a naive unstructured doubled real factorisation has leading scale proportional to

$$(2n)^3 = 8n^3.$$

The same cubic exponent is retained, but the structured CRE representation gives a leading-constant gain in the SPD case and removes the artificial factor associated with treating the block display as four independent real blocks.

Proof. The classical dense Cholesky factorisation has leading count $\frac{1}{3}n^3 + O(n^2)$, while generic LU-type factorisation has leading count $\frac{2}{3}n^3 + O(n^2)$. The storage-neutral CRE representation does not store four independent $n \times n$ blocks; it stores the two independent channels. Therefore the appropriate comparison is between structured channel Cholesky and unstructured doubled real factorisation, not between the original algebra and an arbitrary $2n \times 2n$ real matrix. □

Remark 12.8 (No Subcubic Claim)**.** *The CRE framework does not claim a subcubic dense direct inverse algorithm. The claim is structural and constant-level: by preserving the algebra as a storage-neutral real block representation, CRE makes it possible to use real SPD and Schur-complement kernels without paying the generic unstructured $2n \times 2n$ cost.*

Remark 12.9 (Practical Runtime Considerations)**.** *The arithmetic analysis above concerns asymptotic order and leading-factorisation constants. Practical runtime behaviour may exhibit additional gains that are not captured by arithmetic counts alone.*

In particular, SPD Cholesky factorisation is pivot-free, whereas generic LU-type factorisations require pivot selection and row-permutation operations. These introduce memory-traffic overhead, cache-disruption effects, and implementation-level synchronisation costs that are absent in the pivot-free SPD setting.

Consequently, the observed runtime improvement of a CRE–SPD pipeline may exceed the improvement predicted solely by leading-term arithmetic analysis. This phenomenon was already observed in the AXIOM implementation study [5].

13 CRE and Twisted XOR Algebras

The signed-XOR calculus supplies the algebraic support law behind the ALH and CRE constructions. Let V be an elementary abelian two-group and let

$$e_\alpha e_\beta = \theta(\alpha, \beta) e_{\alpha+\beta}$$

be a normalized signed-XOR algebra.

Definition 13.1 (Twisted XOR Algebra)**.** *The algebra $\mathbb{R}_\theta[V]$ is the real vector space with basis $\{e_\alpha : \alpha \in V\}$ and product*

$$e_\alpha e_\beta = \theta(\alpha, \beta) e_{\alpha+\beta}, \qquad \theta(\alpha, \beta) \in \{\pm 1\}.$$

Theorem 13.2 (CRE Preservation of the XOR Support Law)**.** *Let $\mathbb{R}_\theta[V]$ be a signed-XOR algebra represented by CRE-admissible operators. Then CRE preserves the support law:*

$$\operatorname{supp}\big(\mathrm{CRE}(e_\alpha)\,\mathrm{CRE}(e_\beta)\big) = \alpha + \beta.$$

The sign twist is carried into the real block representation by

$$\mathrm{CRE}(e_\alpha)\,\mathrm{CRE}(e_\beta) = \theta(\alpha, \beta)\,\mathrm{CRE}(e_{\alpha+\beta}).$$

Proof. This follows from multiplicativity of CRE:

$$\mathrm{CRE}(e_\alpha)\,\mathrm{CRE}(e_\beta) = \mathrm{CRE}(e_\alpha e_\beta) = \mathrm{CRE}(\theta(\alpha, \beta) e_{\alpha+\beta}).$$

Since $\theta(\alpha, \beta)$ is a real scalar, it factors out of CRE. The displayed formula follows. □

Theorem 13.3 (Gauge Invariance of CRE Kernels)**.** *Let θ and θ' be gauge-equivalent twists:*

$$\theta'(\alpha, \beta) = \frac{\lambda(\alpha)\lambda(\beta)}{\lambda(\alpha+\beta)} \theta(\alpha, \beta), \qquad \lambda(\alpha) \in \{\pm 1\}.$$

Then the corresponding CRE representations are conjugate by a signed orthogonal change of basis. Consequently their spectra, SPD kernels, and positivity properties coincide.

Proof. Define the signed basis change by

$$e_\alpha \longmapsto \lambda(\alpha) e_\alpha.$$

The gauge relation is precisely the condition that this map intertwines the two twisted products. In the CRE representation this signed basis change is an orthogonal signed-permutation matrix. Orthogonal conjugacy preserves spectra, kernels of the form $F^T F$, and positivity. □

14 CRE versus Classical Realification

A natural question is whether CRE is merely a reformulation of the classical realification procedure. The answer is negative. While both constructions replace complex operators by real block operators, CRE preserves additional operator-theoretic structures that play a central role in positivity analysis and inverse reconstruction.

Theorem 14.1 (Adjoint Preservation)**.** *For every operator A,*

$$\mathrm{CRE}(A^*) = \mathrm{CRE}(A)^T.$$

Hence the adjoint structure is preserved exactly.

Proof. This follows directly from the block representation established in the Faithful Embedding theorem. □

Theorem 14.2 (Kernel Reconstruction Principle)**.** *Let*

$$\mathcal{K}(A) = \mathrm{CRE}(A)^T \,\mathrm{CRE}(A).$$

Then positivity, invertibility, and defect structure of A can be studied through the SPD kernel $\mathcal{K}(A)$.

Proof. The CRE–SPD Reduction theorem converts operator analysis into positive-semidefinite kernel analysis, while the Invertibility Recovery theorem reconstructs the inverse from the kernel whenever strict positivity holds. □

Theorem 14.3 (Algebraic Structure Preservation)**.** *Let $\mathcal{A}$ be a CRE-admissible operator algebra. Then CRE introduces no new algebraic structure.*
The image algebra

$$\mathrm{CRE}(\mathcal{A})$$

inherits its multiplication, involution, spectrum, positivity cone, and invertibility structure from

$$\mathcal{A}.$$

Consequently,

$$\mathcal{A} \cong_* \mathrm{CRE}(\mathcal{A}).$$

Proof. The Faithful Embedding theorem preserves multiplication and adjoints, while the Spectrum Preservation, HPD–SPD Equivalence, and Invertibility Equivalence theorems preserve spectral, positivity, and invertibility structure. The Intrinsic CRE $*$-Isomorphism theorem identifies the original slice-linear algebra with its CRE image. Therefore no new algebraic structure is created; the same structure is represented in real block form. □

Theorem 14.4 (Strict Extension Beyond Realification)**.** *Classical realification provides a real block representation of operators. CRE additionally provides:*

1. *exact adjoint preservation,*
2. *projection-valued defect analysis,*
3. *SPD reduction,*
4. *Cholesky-based inverse reconstruction,*

5. *compatibility with ALH quarter-turn structures.*

Consequently CRE constitutes a strictly richer computational framework for positivity analysis, defect detection, and inverse reconstruction.

Proof. The listed properties follow from the previous sections. In particular, neither projection-valued defect theory nor inverse recovery through SPD kernels is part of the standard realification formalism. Therefore CRE organises the transported information into additional computational structures. □

The conceptual role of CRE may therefore be summarized as

$$\text{Algebra} \longrightarrow \text{CRE Operator} \longrightarrow \text{SPD Kernel} \longrightarrow \text{Inverse Reconstruction}.$$

This progression is the mechanism through which algebraic questions are transformed into positivity and operator-theoretic questions.

15 Defect–Invertibility Correspondence

The previous sections introduced three central notions: projection-valued defects, SPD kernels, and invertibility recovery. We now show that these are equivalent manifestations of the same underlying phenomenon.

Theorem 15.1 (Defect–Invertibility Equivalence)**.** *Let A be a finite-dimensional CRE-admissible operator and let*

$$\mathcal{K}(A) = \mathrm{CRE}(A)^T \,\mathrm{CRE}(A).$$

Then the following are equivalent:

1. *$\mathcal{K}(A) \succ 0$;*
2. *A is invertible;*
3. *there exists no nontrivial projection-valued defect.*

Proof. By the Positivity Obstruction theorem, strict positivity of $\mathcal{K}(A)$ is equivalent to the absence of nontrivial projection-valued defects contained in its null space.

In the present finite-dimensional setting, every failure of strict positivity produces a nontrivial spectral projection onto the zero eigenspace, and every such projection is a projection-valued defect. Hence (1) and (3) are equivalent.

By the Invertibility Recovery theorem, strict positivity of $\mathcal{K}(A)$ implies invertibility of A. Conversely, if A is invertible, then $\mathrm{CRE}(A)$ is invertible by the Invertibility Equivalence theorem. Therefore

$$\mathcal{K}(A) = \mathrm{CRE}(A)^T \,\mathrm{CRE}(A)$$

has trivial kernel and is strictly positive definite. Thus (1) and (2) are equivalent.

Combining these implications proves the theorem. □

Corollary 15.2 (Unified CRE Criterion)**.** *For CRE-admissible operators, positivity, defect-freeness, and invertibility are equivalent conditions.*

Proof. This is an immediate consequence of the Defect–Invertibility Equivalence theorem. □

The preceding theorem is the structural core of the CRE framework. Projection-valued defects encode the failure of positivity, positivity controls invertibility, and invertibility is reconstructed through the SPD kernel. Consequently the chain

$$\text{Defect Theory} \Longleftrightarrow \text{SPD Positivity} \Longleftrightarrow \text{Invertibility}$$

forms a single unified structure.

16 Structural Synthesis

The constructions above combine into the structural chain

$$\begin{aligned}\text{Signed-XOR} &\Longrightarrow \text{Twisted Group Algebra} \Longrightarrow \text{ALH Slice Structure} \Longrightarrow \text{CRE} \\ &\Longrightarrow \text{SPD Kernel / NGC-Regularised SPD Channel} \Longrightarrow \text{Defect Theory} \\ &\Longrightarrow \text{Invertibility Reconstruction} \Longrightarrow \text{Storage-Neutral Block Computation.}\end{aligned}$$

Each arrow has a precise meaning. Signed-XOR supplies the support law and twist. ALH extracts local complex slices and quarter-turn atlases, providing the slice-level algebraic structures on which CRE acts faithfully. The rotation chart and null-gauge mapping may be chosen flexibly, but reconstruction must use the same chart used for the forward CRE embedding. CRE turns those slices, including quaternionic, octonionic, sedenionic, and higher Cayley–Dickson ALH slices, into faithful real block operators. Consequently the same CRE–SPD inverse-recovery mechanism applies not only to complex operators, but also to general tensor operators over quaternionic, octonionic, sedenionic, and higher Cayley–Dickson ALH slices. The SPD kernel converts operator invertibility into positivity; when the symmetric CRE core is indefinite, NGC transfers the computation to an SPD compensated channel without altering the genuine kernel. Projection-valued defects classify the failure of strict positivity. Cholesky reconstruction then recovers the inverse whenever no structural defect remains. Because CRE stores symmetric and skew channels rather than four independent blocks, the apparent dimensional doubling is a coordinate display rather than a doubling of algebraic information. This is what makes Schur-complement and structured SPD inverse pathways meaningful for complexity analysis.

Theorem 16.1 (CRE Structural Synthesis)**.** *For finite-dimensional CRE-admissible operators arising from signed-XOR or ALH data, the following data are equivalent ways of detecting the same obstruction:*

1. *nontrivial projection-valued defect;*
2. *failure of strict SPD positivity of the CRE kernel;*
3. *failure of invertibility;*
4. *obstruction to Cholesky inverse reconstruction.*

Proof. The equivalence between the first three items is the Defect–Invertibility Equivalence theorem. The equivalence with the fourth follows from the Cholesky Reconstruction theorem: a Cholesky inverse reconstruction exists precisely when the CRE kernel is SPD. □

17 Outlook

Future work includes computational implementations of CRE–SPD inverse recovery, structured Schur-complement solvers for Cayley–Dickson tensor operators, large-scale benchmark studies of pivot-free SPD reconstruction, Hilbert–Pólya operators, null-gauge spectral theory, and arithmetic-local operator geometry. The present work establishes the algebraic and operator-theoretic foundation on which those computational and spectral developments may be built.

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