ARITHMETIC LOCAL HURWITZ SLICES AND QUARTER-TURN ACTIVATION OVER ARBITRARY INDEX SYSTEMS

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Abstract. Arithmetic Local Hurwitz (ALH) is formulated as a self-contained local-slice theorem and then extended to a quarter-turn activation interface. The core assertion is that, in any real unital $*$-algebraic system, every imaginary generator e_i satisfying

$$e_i^2 = -1, \qquad e_i^* = -e_i$$

spans a two-dimensional real $*$-subalgebraic slice

$$C_i = \mathbb{R}1 \oplus \mathbb{R}e_i$$

canonically $*$-isomorphic to $\mathbb{C}$. Consequently inversion, conjugation, and norm composition hold on each local slice, independently of the ambient dimension or of whether the index labels are finite, countable, integer-valued, real-valued, or complex-valued. Labels name generators; they do not by themselves insert the arithmetic of the label set into the algebra.

A signed-XOR finite-support backend is included to place these local slices in table-free ambient algebras. In the sorted Clifford sector,

$$e_J e_K = (-1)^{\mathrm{inv}(J,K)+|J\cap K|} e_{J\triangle K},$$

while more general twists keep the same XOR support law and record associativity failure by an explicit associator sign. Thus ALH is separated from global Hurwitz-composition or division claims.

The interface layer applies the same theorem to quarter-turn operators J with $J^2 = -I_E$ and $J^* = -J$ on the relevant real space E. It yields real and complex activation maps inside the local slice, regular orthogonal-coordinate realizations, and CRE-type liftings. Complex activation parameters are phase–scale parameters, not cardinal dimensions. A total activation $+1$ gives the oriented generator, while a total activation -1 gives the reversed orientation. Every admissible CRE-type lifting is therefore a $*$-isomorphic realization of the same local ALH slice; real block models are one display convention, not the whole scope of the theorem.

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1. Introduction

The purpose of this paper is twofold. The first purpose is to isolate Arithmetic Local Hurwitz (ALH) as an independent mathematical structure. The second is to record the quarter-turn activation and CRE interfaces that follow from the same local theorem. In the AXIOM thesis [4], ALH appears as one of the structural conditions used to support the Conjugation–Real Embedding (CRE): each generator produces a complex slice, and compatible finite slices provide the local arithmetic needed for inversion and real block representation. That placement is natural for a thesis-level exposition, but it hides the fact that ALH is itself a complete local theorem.

The local theorem is the following. If A is a real unital $*$-algebraic system and $e_i \in A$ satisfies

$$e_i^2 = -1, \qquad e_i^* = -e_i,$$

then

$$C_i = \operatorname{span}_{\mathbb{R}}\{1, e_i\}$$

is a copy of the complex numbers. The map

$$\phi_i : C_i \to \mathbb{C}, \qquad \phi_i(a + be_i) = a + bi$$

preserves addition, multiplication, the unit, and the $*$-operation. Hence

$$(a + be_i)^{-1} = \frac{a - be_i}{a^2 + b^2}, \qquad N_i((a + be_i)(c + de_i)) = N_i(a + be_i)N_i(c + de_i),$$

where $N_i(a + be_i) = a^2 + b^2$.

This is the ALH core. It is local, functorial under reindexing, and insensitive to the cardinality or scalar nature of the label set. The label i may be drawn from $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$, or any other indexing set. It remains only a label unless additional structure is explicitly imposed. In particular, using $\mathbb{C}$ as an index set does not mean that the field operations of $\mathbb{C}$ are inserted into the multiplication of the algebra. It means that the family of imaginary directions is labelled by complex numbers.

The paper also clarifies what ALH does not claim. It does not assert the existence of global normed division algebras in arbitrary dimension. Hurwitz's theorem and the theory of composition algebras remain intact [3, 9]. ALH replaces a global composition requirement by a slice-wise one. The local complex plane C_i is always a normed division algebra; the ambient algebra may be noncommutative, nonassociative, or globally nondivision.

The paper should therefore be read as an ALH theorem paper with a second interface layer. After the core theorem and its signed-XOR finite-support realization are established, the paper records several interfaces. The curl activation, complex activation, orthogonal-coordinate, and CRE-type lifting sections are applications of the local theorem, not additional assumptions needed to prove it. This separation is intentional: the main theorem is the local $*$-isomorphism $C_i \cong_{\mathbb{R},*} \mathbb{C}$, while the later sections show how that theorem is used in activation, coordinate, and operator representations.

Main contributions. The results are organized around a core theorem and a set of interfaces.

First, ALH is stated as a local $*$-isomorphism theorem. Each generator e_i with $e_i^2 = -1$ and $e_i^* = -e_i$ gives a canonical copy of $\mathbb{C}$.

Second, arbitrary index systems are handled by finite support. Every product involves only finitely many labels, so infinite or uncountable label sets are filtered by finite subsystems.

Third, the signed-XOR backend is included. In the sorted Clifford sector, basis products are computed by symmetric difference and a parity sign. In a general signed-XOR twist, the same support law persists while associativity is controlled by an explicit associator sign. This follows the support/sign separation developed in [5].

Fourth, orientation reversal is internal to ALH. Replacing e_i by $-e_i$ gives the same unoriented imaginary axis and the same slice, while the local complex chart is conjugated.

Fifth, the interface sections prove that the local theorem is stable under real and complex activation sums, regular orthogonal coordinate frames, and admissible CRE-type liftings. Both targets $+1$ and -1 are valid in the activation calculus: a total activation $+1$ produces e_i, while a total activation -1 produces the orientation-reversed generator $-e_i$.

Logical status.

Status Note 1.1. This paper proves local algebraic statements. It does not prove a global Hurwitz theorem or a global division theorem. It supplies the local complex-slice calculus used by CRE-type block and operator realizations. Prior operator-framework uses of CRE-type liftings may therefore be cited here only for their local representation mechanism, not for any later positivity, determinant, or arithmetic input. Analytic questions such as positivity, conditioning, spectral comparison, or arithmetic construction are outside the scope of the present ALH theorem.

Notation and standing conventions. We use I only for an index or label set. Identity operators are written with subscripts: I_E on a real space E, I_H on a Hilbert block H, and I_P or $I_{P_{ab}}$ on an oriented coordinate two-plane. This avoids confusing the identity operator with the label set I.

The symbol $*$ denotes the relevant involution or adjoint. In an abstract real $*$-algebraic system it is the algebra involution. On a Hilbert space or coordinate two-plane it is the Hilbert-space adjoint or transpose with respect to the chosen inner product. Under a local ALH chart it corresponds to ordinary complex conjugation.

A slice means a two-dimensional real algebraic plane of the form

$$\mathbb{R}1 \oplus \mathbb{R}e_i \qquad \text{or} \qquad \{aI_E + bJ : a, b \in \mathbb{R}\}.$$

Thus a slice is a local algebraic copy of $\mathbb{C}$, not a coordinate chart on a manifold. The coordinate sections later show how normalized orthogonal coordinate planes produce such algebraic slices.

The words "activation" and "curl" are used only for the normalized quarter-turn exponential inside a local ALH slice. They do not refer to fractional vector-space dimension or to the vector-calculus curl operator. Real and complex activation parameters are therefore parameters of a local phase–scale calculus, not changes in vector-space cardinality.

2. Basic algebraic setting

2.1. Real $*$-algebras and imaginary generators.

Definition 2.1 (Real unital $*$-algebraic system)**.** A real unital $*$-algebraic system is a real unital algebraic system A, not necessarily associative unless stated, equipped with an $\mathbb{R}$-linear map $* : A \to A$ such that

$$(xy)^* = y^*x^*, \qquad (x^*)^* = x,$$

for all products xy under consideration, and $1^* = 1$. When associativity is needed globally, we explicitly say that A is associative. The local ALH slice generated by one element is associative by reduction to $\mathbb{R}1 \oplus \mathbb{R}e_i$.

Definition 2.2 (ALH generator)**.** Let A be a real unital $*$-algebraic system. An element $e \in A$ is an ALH imaginary generator if

$$e^2 = -1, \qquad e^* = -e.$$

A family $\{e_i : i \in I\}$ of such elements is an ALH indexed family.

The definition deliberately imposes no global associativity beyond the products used in the local slice. When A is nonassociative, the local statements below are read inside the subalgebra generated by one element e_i, where every product reduces to a real linear combination of 1 and e_i using the single relation $e_i^2 = -1$. Hence the local slice is associative even if the ambient algebra is not. In the signed-XOR and Clifford sectors used later, associativity is available in the sorted sector; in the Cayley–Dickson sector, local complex slices remain associative even when the ambient algebra is not.

Definition 2.3 (Local ALH slice)**.** For an ALH generator e_i, define

$$C_i = \mathbb{R}1 \oplus \mathbb{R}e_i.$$

The local conjugation on C_i is the restriction of $*$:

$$(a + be_i)^* = a - be_i.$$

The local norm is

$$N_i(a + be_i) = (a + be_i)^*(a + be_i) = a^2 + b^2.$$

2.2. **The local complex slice theorem.**

Theorem 2.4 (ALH local complex $*$-isomorphism)**.** *Let A be a real unital $*$-algebraic system and let $e_i \in A$ satisfy $e_i^2 = -1$ and $e_i^* = -e_i$. Then*

$$C_i = \mathbb{R}1 \oplus \mathbb{R}e_i$$

is a local unital real $*$-subalgebraic slice of A, associative with the displayed multiplication. Moreover

$$\phi_i : C_i \to \mathbb{C}, \qquad \phi_i(a + be_i) = a + bi,$$

is a unital real $*$-algebra isomorphism, where $*$ on $\mathbb{C}$ is complex conjugation.

Proof. Let $x = a + be_i$ and $y = c + de_i$. Using $e_i^2 = -1$,

$$xy = (a + be_i)(c + de_i) = (ac - bd) + (ad + bc)e_i \in C_i.$$

Thus C_i is closed under multiplication and contains 1. The map ϕ_i is real-linear and bijective. The displayed multiplication gives

$$\phi_i(xy) = (ac - bd) + (ad + bc)i = (a + bi)(c + di) = \phi_i(x)\phi_i(y).$$

Also

$$\phi_i(x^*) = \phi_i(a - be_i) = a - bi = \overline{a + bi} = \overline{\phi_i(x)}.$$

Hence ϕ_i is a unital real $*$-algebra isomorphism. □

Corollary 2.5 (Local inverse and norm composition)**.** *For $x = a + be_i \in C_i$ with $(a, b) \neq (0, 0)$,*

$$x^{-1} = \frac{x^*}{N_i(x)} = \frac{a - be_i}{a^2 + b^2}.$$

For all $x, y \in C_i$,

$$N_i(xy) = N_i(x)N_i(y).$$

Proof. Since ϕ_i is a $*$-isomorphism, the statements are the pullbacks of the usual inverse and modulus identities in $\mathbb{C}$. Directly,

$$(a + be_i)(a - be_i) = a^2 + b^2.$$

The norm composition identity follows either from Theorem 2.4 or from the identity $|zw|^2 = |z|^2|w|^2$ in $\mathbb{C}$. □

Remark 2.6 (Minimality of the ALH hypothesis). The local theorem uses only two equations: $e_i^2 = -1$ and $e_i^* = -e_i$. It does not use the cardinality of I, any ordering on I, or any global multiplication table. These additional structures enter only when several generators are multiplied.

3. Finite supports and signed-XOR realization

The previous section proves ALH in its minimal form. This section records a finite-support backend that is useful when one wants a table-free ambient algebra containing many local slices. The presentation is self-contained, but it follows the support/sign separation of signed-XOR structural constants [5]. Classical Clifford and Cayley–Dickson references include [1, 2, 6–8].

3.1. **Finite-support elementary two-groups.**

Definition 3.1 (Finite-support group)**.** Let I be an index set. Define

$$\mathrm{Fin}(I) = \{J \subset I : |J| < \infty\}.$$

The group operation is symmetric difference

$$J \triangle K = (J \setminus K) \cup (K \setminus J).$$

Equivalently, $\mathrm{Fin}(I)$ is the elementary abelian two-group of finite-support maps $I \to \mathbb{F}_2$.

Finite support is the mechanism that makes arbitrary label systems harmless. Even if I is uncountable, each basis monomial uses only finitely many labels. Thus every displayed product is computed in a finite subsystem before passing back to the ambient indexed family.

When I is linearly ordered and $J = \{j_1 < \cdots < j_m\}$, write

$$e_J = e_{j_1} \cdots e_{j_m}, \qquad e_\varnothing = 1.$$

For finite subsets $J, K \subset I$, set

$$\mathrm{inv}(J, K) = \#\{(j, k) \in J \times K : j > k\}.$$

3.2. **The sorted Clifford sector.**

Definition 3.2 (Sorted Clifford sector)**.** Let $(I, <)$ be a linearly ordered set. The sorted Clifford sector $A_{\mathrm{sort}}(I)$ is the unital real associative algebra generated by $\{e_i : i \in I\}$ with relations

$$e_i^2 = -1, \qquad e_i e_j = -e_j e_i \quad (i \neq j).$$

The algebra can be defined for arbitrary I because every word contains only finitely many generators. If one wants a sorted basis, a linear order is convenient. For $I = \mathbb{N}, \mathbb{Z}, \mathbb{R}$ the usual order may be used; for $I = \mathbb{C}$ one may fix, for example, the lexicographic order on $(\mathrm{Re}\, z, \mathrm{Im}\, z)$. The local ALH theorem itself is independent of this choice.

Theorem 3.3 (Sorted normal form)**.** *The elements $\{e_J : J \in \mathrm{Fin}(I)\}$ form an $\mathbb{R}$-basis of $A_{\mathrm{sort}}(I)$. Every monomial in the generators reduces uniquely to a signed sorted monomial $\pm e_J$.*

Proof. The defining relations allow every word to be reduced by repeatedly replacing $e_i e_i$ by -1 and swapping adjacent out-of-order generators at the cost of a minus sign. The process terminates because either the word length decreases or the inversion number of the word decreases.

To prove linear independence, use the standard signed-permutation model. Let E_I be the real vector space with basis $\{u_J : J \in \mathrm{Fin}(I)\}$. For $i \in I$, define

$$L_i u_J = (-1)^{\#\{j \in J : j < i\} + \mathbf{1}_{i \in J}} u_{J \triangle \{i\}}.$$

A direct check gives $L_i^2 = -\,\mathrm{id}$ and $L_i L_j = -L_j L_i$ for $i \neq j$. By the universal property, there is a representation $\rho : A_{\mathrm{sort}}(I) \to \mathrm{End}_{\mathbb{R}}(E_I)$ with $\rho(e_i) = L_i$.

If $J = \{j_1 < \cdots < j_m\}$, then $\rho(e_J)u_\varnothing = \pm u_J$. A linear relation among distinct e_J would therefore give a linear relation among distinct basis vectors u_J, impossible. Thus the sorted monomials are linearly independent and form a basis. □

Theorem 3.4 (Signed-XOR product formula)**.** *For all $J, K \in \mathrm{Fin}(I)$,*

$$e_J e_K = (-1)^{\mathrm{inv}(J,K)+|J\cap K|} e_{J\triangle K}.$$

Proof. In the concatenated word $e_J e_K$, every pair $(j,k) \in J \times K$ with $j > k$ must be swapped once to reach sorted order, giving the sign $(-1)^{\mathrm{inv}(J,K)}$. Each common index then contributes one square $e_i^2 = -1$, giving the additional sign $(-1)^{|J\cap K|}$. The remaining support is exactly the symmetric difference $J\triangle K$. □

Corollary 3.5 (Clifford identification)**.** *If I is finite with $|I| = n$, then*

$$A_{\mathrm{sort}}(I) \cong Cl_{0,n}(\mathbb{R}).$$

If I is infinite, then $A_{\mathrm{sort}}(I)$ is the filtered direct limit of its finite Clifford subalgebras.

Proof. For finite I, the generators and relations are exactly the standard presentation of $Cl_{0,n}(\mathbb{R})$. For infinite I, every basis element and every finite product uses only a finite subset $F \subset I$, hence lies in the finite subalgebra $A_{\mathrm{sort}}(F) \cong Cl_{0,|F|}(\mathbb{R})$. □

3.3. **General signed-XOR twists.**

Definition 3.6 (Twisted signed-XOR algebra)**.** Let V be an elementary abelian two-group and let

$$\theta : V \times V \to \{\pm 1\}$$

be normalized, meaning $\theta(0,\alpha) = \theta(\alpha,0) = 1$. Define $\mathbb{R}_\theta[V]$ to be the real vector space with basis $\{e_\alpha : \alpha \in V\}$ and product

$$e_\alpha e_\beta = \theta(\alpha,\beta) e_{\alpha+\beta}.$$

The sorted Clifford sector is the special case $V = \mathrm{Fin}(I)$ and

$$\theta_{\mathrm{sort}}(J,K) = (-1)^{\mathrm{inv}(J,K)+|J\cap K|}.$$

Cayley–Dickson basis multiplication can also be expressed in this form after replacing the sorted Clifford cocycle by the recursive Cayley–Dickson sign twist; the support update remains XOR while the associator changes [5].

Proposition 3.7 (Associator sign)**.** *In $\mathbb{R}_\theta[V]$,*

$$(e_\alpha e_\beta)e_\gamma = \omega_\theta(\alpha,\beta,\gamma) e_\alpha (e_\beta e_\gamma),$$

where

$$\omega_\theta(\alpha,\beta,\gamma) = \frac{\theta(\alpha,\beta)\theta(\alpha+\beta,\gamma)}{\theta(\beta,\gamma)\theta(\alpha,\beta+\gamma)}.$$

The algebra is associative if and only if $\omega_\theta \equiv 1$.

Proof. Both parenthesizations are scalar multiples of $e_{\alpha+\beta+\gamma}$. Comparing the scalar coefficients gives the displayed formula. Associativity is exactly equality of the two scalar coefficients for all triples. □

Proposition 3.8 (Local ALH inside a signed-XOR algebra)**.** *Let $\mathbb{R}_\theta[V]$ be a signed-XOR algebra. Suppose $\delta \in V$ satisfies*

$$\theta(\delta, \delta) = -1.$$

Then $e_\delta^2 = -1$, and

$$C_\delta = \mathbb{R}1 \oplus \mathbb{R}e_\delta$$

is a local complex ALH slice. If a $*$-operation is given with $e_\delta^* = -e_\delta$, then the slice is $*$-isomorphic to $\mathbb{C}$.

Proof. Since $\delta + \delta = 0$ in the elementary two-group,

$$e_\delta^2 = \theta(\delta, \delta)e_0 = -1.$$

The rest is Theorem 2.4. □

4. Arithmetic Local Hurwitz structures

4.1. Definition of ALH.

Definition 4.1 (Arithmetic Local Hurwitz structure)**.** An Arithmetic Local Hurwitz structure is a tuple

$$(A, *, I, \{e_i\}_{i\in I})$$

where A is a real unital $*$-algebraic system and each e_i is an ALH imaginary generator:

$$e_i^2 = -1, \qquad e_i^* = -e_i.$$

The associated local complex slices are

$$C_i = \mathbb{R}1 \oplus \mathbb{R}e_i.$$

If specified, a compatible finite slice is a finite-dimensional subalgebra generated by finitely many e_i on which the intended associative, alternative, or composition identities are known to hold.

The word "local" is essential. ALH consists of local complex arithmetic at each generator, together with optional compatibility data for finite multi-generator slices. It is not a global composition condition on all of A.

Theorem 4.2 (ALH package)**.** *Let $(A, *, I, \{e_i\})$ be an ALH structure. Then for each $i \in I$:*

(a) C_i is a local unital real $*$-subalgebraic slice of A;
(b) $C_i \cong_{\mathbb{R},*} \mathbb{C}$ by $a + be_i \mapsto a + bi$;
(c) every nonzero $x \in C_i$ has inverse $x^{-1} = x^*/N_i(x)$;
(d) $N_i(xy) = N_i(x)N_i(y)$ for all $x, y \in C_i$.

Proof. This is Theorem 2.4 and Corollary 2.5 applied to each $i \in I$. □

4.2. **Quaternionic compatible slices.**

Definition 4.3 (Associative anticommuting pair)**.** Two ALH generators e_i, e_j form an associative anticommuting pair if the subalgebra generated by e_i, e_j is associative and

$$e_i e_j = -e_j e_i.$$

Theorem 4.4 (Quaternionic slice)**.** *Let e_i, e_j form an associative anticommuting pair. Then the assignment*

$$\mathbf{i} \longmapsto e_i, \qquad \mathbf{j} \longmapsto e_j, \qquad \mathbf{k} \longmapsto e_i e_j$$

defines a surjective real algebra homomorphism

$$\mathbb{H} \longrightarrow H_{ij} := \operatorname{span}_{\mathbb{R}}\{1, e_i, e_j, e_i e_j\}.$$

If, in addition, the four elements $1, e_i, e_j, e_i e_j$ are linearly independent over $\mathbb{R}$, then this homomorphism is an isomorphism and $H_{ij} \cong \mathbb{H}$.

Proof. Let $k = e_i e_j$. By associativity and anticommutation,

$$k^2 = e_i e_j e_i e_j = e_i(-e_i e_j)e_j = -e_i^2 e_j^2 = -(-1)(-1) = -1.$$

Also $e_i e_j = k$, $e_j e_i = -k$, and the remaining quaternionic identities follow by associativity. Hence the universal presentation of $\mathbb{H}$ gives a homomorphism from $\mathbb{H}$ onto the subalgebra generated by e_i and e_j, whose real span is H_{ij}. If $1, e_i, e_j, e_i e_j$ are linearly independent, the target has real dimension four, so the surjective homomorphism from the four-dimensional algebra $\mathbb{H}$ is an isomorphism. □

Corollary 4.5 (Sorted two-generator slices)**.** *In $A_{\mathrm{sort}}(I)$, every pair $i \neq j$ gives a quaternionic slice $H_{ij} \cong \mathbb{H}$.*

Proof. The sorted Clifford sector is associative and its distinct generators anticommute by definition. By the normal-form theorem, the elements $1, e_i, e_j, e_i e_j$ are linearly independent. Apply Theorem 4.4. □

4.3. **No conflict with Hurwitz.**

Proposition 4.6 (ALH is not a global Hurwitz theorem)**.** *ALH does not assert that the ambient algebra A is a normed division algebra or that a scalar norm satisfies $N(xy) = N(x)N(y)$ for all $x, y \in A$. It asserts only that each C_i is a local copy of $\mathbb{C}$, and that compatible finite slices have whatever additional composition properties are explicitly assumed or proved. Therefore ALH is compatible with the classical Hurwitz obstruction for finite-dimensional real normed division algebras.*

Proof. The ALH theorem is restricted to C_i, where $C_i \cong \mathbb{C}$ is two-dimensional. Hurwitz's theorem constrains global finite-dimensional normed division algebras with multiplicative quadratic norm. A family of two-dimensional local slices inside a larger algebra is not such a global assertion. Hence there is no logical conflict. □

Remark 4.7. This point is not cosmetic. If one writes "norm composition holds in all dimensions" without the local qualifier, the statement is false. The correct statement is: norm composition holds on each ALH complex slice, and on explicitly compatible finite Hurwitz slices.

5. Arbitrary index systems and reindexing

5.1. **Labels versus algebraic structure.**

Definition 5.1 (Label system)**.** A label system is a set I used to name the distinguished generators $\{e_i : i \in I\}$. Unless additional structure is specified, the elements of I carry no algebraic operations inside A.

Thus $I = \mathbb{C}$ is allowed, but $z + w$ or zw in the label set does not automatically determine $e_z e_w$. The multiplication of generators is defined by the chosen ambient algebra, for example by the sorted Clifford relations or by a signed-XOR twist.

Proposition 5.2 (Finite locality over arbitrary labels)**.** *Let I be any set and let A be generated by $\{e_i : i \in I\}$ subject to finite-word relations. Every element of the algebraic span has finite support: it lies in the subalgebra generated by $\{e_i : i \in F\}$ for some finite $F \subset I$. Consequently every identity involving finitely many elements is checked inside a finite subsystem.*

Proof. Each monomial contains finitely many symbols. A finite linear combination of monomials therefore uses only the union of finitely many finite supports, hence a finite subset $F \subset I$. □

5.2. **Naturality under reindexing.**

Theorem 5.3 (Reindexing naturality of ALH)**.** *Let*

$$(A, *, I, \{e_i\}_{i\in I}) \quad \textit{and} \quad (A', *', I', \{e'_j\}_{j\in I'})$$

be ALH structures. Suppose $f : I \to I'$ is a map and $\Psi : A \to A'$ is a unital $*$-homomorphism such that

$$\Psi(e_i) = e'_{f(i)}$$

for every $i \in I$. Then

$$\Psi(C_i) \subset C'_{f(i)}$$

and the diagram

$$\begin{array}{ccc} C_i & \xrightarrow{\Psi} & C'_{f(i)} \\ \phi_i \downarrow & & \downarrow \phi'_{f(i)} \\ \mathbb{C} & \xrightarrow{\mathrm{id}} & \mathbb{C} \end{array}$$

commutes.

Proof. For $a, b \in \mathbb{R}$,

$$\Psi(a + be_i) = a + b\Psi(e_i) = a + be'_{f(i)} \in C'_{f(i)}.$$

Applying the two slice charts gives

$$\phi'_{f(i)}(\Psi(a + be_i)) = a + bi = \phi_i(a + be_i).$$

Thus the diagram commutes. □

Corollary 5.4 (Sorted Clifford functoriality)**.** *If $f : I \hookrightarrow I'$ is an injection, then the assignment $e_i \mapsto e'_{f(i)}$ extends uniquely to a unital algebra monomorphism*

$$A_{\mathrm{sort}}(I) \hookrightarrow A_{\mathrm{sort}}(I').$$

It carries each local slice C_i onto $C'_{f(i)}$ and preserves the ALH chart.

Proof. The images satisfy the same relations:

$$(e'_{f(i)})^2 = -1, \qquad e'_{f(i)}e'_{f(j)} = -e'_{f(j)}e'_{f(i)} \quad (i \neq j).$$

The universal property gives the homomorphism. Injectivity follows from the normal-form basis, because distinct finite supports in I map to distinct finite supports in I'. The slice statement follows from Theorem 5.3. □

Remark 5.5 (Complex-valued labels). For $I = \mathbb{C}$, a convenient order for the sorted formula is lexicographic order on $(\operatorname{Re} z, \operatorname{Im} z)$. This is only a bookkeeping choice for signs in sorted normal forms. The local ALH slice $C_z = \mathbb{R}1 \oplus \mathbb{R}e_z$ and its $*$-isomorphism with $\mathbb{C}$ do not depend on the field structure of the label z.

6. Orientation, half-turns, and phase lifts

6.1. **Oriented and unoriented imaginary directions.** An ALH generator is oriented: e_i and $-e_i$ point in opposite directions along the same imaginary axis. The axis itself is unoriented.

Definition 6.1 (Orientation reversal)**.** Given an ALH generator e_i, define the reversed generator

$$e_{\bar{i}} := -e_i.$$

This notation denotes the same underlying label with the opposite orientation; it is not a new independent generator.

Theorem 6.2 (Orientation reversal preserves ALH)**.** *Let e_i be an ALH generator and set $e_{\bar{i}} = -e_i$. Then $e_{\bar{i}}$ is also an ALH generator, and*

$$C_{\bar{i}} = \mathbb{R}1 \oplus \mathbb{R}e_{\bar{i}} = \mathbb{R}1 \oplus \mathbb{R}e_i = C_i.$$

The two local charts differ by complex conjugation.

Proof. First,

$$e_{\bar{i}}^2 = (-e_i)^2 = e_i^2 = -1.$$

Also

$$e_{\bar{i}}^* = (-e_i)^* = -e_i^* = e_i = -e_{\bar{i}}.$$

Hence $e_{\bar{i}}$ is an ALH generator. The equality of slices is immediate from $\mathbb{R}(-e_i) = \mathbb{R}e_i$.

Let $\phi_i(a + be_i) = a + bi$ and $\phi_{\bar{i}}(a + be_{\bar{i}}) = a + bi$. If $x = a + be_i$, then $x = a - be_{\bar{i}}$, so

$$\phi_{\bar{i}}(x) = a - bi = \overline{\phi_i(x)}.$$

Thus the charts differ by conjugation. □

6.2. **Half-turn consistency.** A rotation by π sends e_i to $-e_i$. This does not make the imaginary axis undefined. It reverses orientation while preserving the underlying slice.

Corollary 6.3 (Imaginary-axis half-turn consistency)**.** *The unoriented imaginary axis determined by e_i is*

$$\mathbb{R}e_i = \mathbb{R}(-e_i).$$

A half-turn by π changes the oriented generator from e_i to $-e_i$, but the ALH slice remains C_i.

Proof. This is the equality $C_{\bar{i}} = C_i$ from Theorem 6.2 restricted to the imaginary line. □

6.3. **Integer phase lifts.**

Definition 6.4 (Phase-lifted orientation label)**.** A phase-lifted orientation label over a base label p is a pair (p, k) with $k \in \mathbb{Z}$. Under the half-turn convention, define

$$e_{(p,k)} = (-1)^k e_{(p,0)}.$$

Thus increasing k by one represents a π-rotation of the oriented imaginary unit. Increasing k by two returns to the same oriented unit.

Proposition 6.5 (Integer phase shifts are internal)**.** *For every $k \in \mathbb{Z}$,*

$$C_{(p,k)} = C_{(p,0)}.$$

If k is even, the ALH chart agrees with the original chart. If k is odd, the chart is conjugated.

Proof. Since $e_{(p,k)} = (-1)^k e_{(p,0)}$, the real span of 1 and the generator is unchanged. Even k gives the same generator; odd k gives the reversed generator, handled by Theorem 6.2. □

Remark 6.6. This is the precise ALH interpretation of negative direction: if the chosen curl direction is positive, then the opposite curl direction is represented by $-e_i$. It is not undefined; it is the conjugate local chart on the same slice.

7. The quarter-turn bridge

The previous sections contain the algebraic core of ALH. The remaining sections record consequences and interfaces. They may be read independently of the core proof: each one is obtained by applying the local $*$-isomorphism theorem to a chosen skew-adjoint quarter-turn.

The logical bridge is the following simple abstraction. ALH does not require a coordinate system, a realification convention, or a global multiplication table. It requires only a real two-plane, or a real Hilbert space block, carrying an operator J with

$$J^2 = -I_E, \qquad J^* = -J,$$

where I_E denotes the identity on the underlying real space E. Once such a J is chosen, the algebra generated by I_E and J is a local complex slice. Curl activation, complex activation, orthogonal-coordinate quarter-turns, and CRE-type liftings are all instances of this same bridge. The bridge is local and algebraic; it is not limited to the particular real block convention used to display complex matrices.

The interface sections use the same local object in different languages. We isolate it once to avoid any gap between the abstract ALH theorem and the later coordinate or CRE constructions.

Definition 7.1 (Admissible quarter-turn)**.** Let E be a real inner-product space or a real Hilbert space. An operator J on E is an admissible quarter-turn if

$$J^2 = -I_E, \qquad J^* = -J,$$

where I_E is the identity on E. The associated real two-generator algebra is

$$C_J = \{aI_E + bJ : a, b \in \mathbb{R}\}.$$

Theorem 7.2 (Quarter-turn bridge)**.** *Let J be an admissible quarter-turn. Then C_J is a unital real $*$-subalgebra of $B(E)$ and*

$$\phi_J : C_J \to \mathbb{C}, \qquad \phi_J(aI_E + bJ) = a + bi,$$

is a unital real $*$-algebra isomorphism. In particular,

$$(aI_E + bJ)^{-1} = \frac{aI_E - bJ}{a^2 + b^2} \qquad ((a, b) \neq (0, 0)),$$

and, for the slice norm

$$N_J(aI_E + bJ) := a^2 + b^2,$$

one has

$$N_J((aI_E + bJ)(cI_E + dJ)) = N_J(aI_E + bJ)N_J(cI_E + dJ).$$

Proof. This is Theorem 2.4 applied to the real $*$-algebra $B(E)$ and to the generator J. Explicitly,

$$(aI_E + bJ)(cI_E + dJ) = (ac - bd)I_E + (ad + bc)J$$

because $J^2 = -I_E$, and

$$(aI_E + bJ)^* = aI_E - bJ$$

because $J^* = -J$. □

Corollary 7.3 (Transport of ALH across admissible realizations)**.** *Let J_1 and J_2 be admissible quarter-turns on real Hilbert spaces E_1 and E_2. If an isometry $U : E_1 \to E_2$ satisfies*

$$J_2 = UJ_1U^*,$$

then conjugation by U gives a real $*$-isomorphism

$$C_{J_1} \longrightarrow C_{J_2}, \qquad aI_{E_1} + bJ_1 \longmapsto aI_{E_2} + bJ_2,$$

and the local complex charts commute with the identity map of $\mathbb{C}$. If $J_2 = -UJ_1U^*$, then the same conjugation gives the conjugate chart:

$$aI_{E_1} + bJ_1 \longmapsto aI_{E_2} - bJ_2,$$

which corresponds to complex conjugation.

Proof. The first statement follows by direct conjugation:

$$U(aI_{E_1} + bJ_1)U^* = aI_{E_2} + bJ_2.$$

The product and $*$-structure are preserved because conjugation by an isometry is a $*$-isomorphism. If the orientation is reversed, J is replaced by $-J$, and Theorem 6.2 identifies this with the conjugate local chart. □

Remark 7.4 (What the bridge proves). The bridge proves that all later realizations have the same algebraic content once their quarter-turns satisfy $J^2 = -I_E$ and $J^* = -J$ on the relevant real space E. Coordinate choices, frame choices, and CRE-type display conventions may change the representative J, but they do not change the underlying local $*$-isomorphism class.

8. Curl activation interface

8.1. **The real curl map.**

Definition 8.1 (Real curl activation)**.** Let C_i be an ALH local slice. For $t \in \mathbb{R}$, define

$$\mathrm{Curl}_i(t) = \exp\left(\frac{\pi t}{2} e_i\right) := \cos\frac{\pi t}{2} + e_i \sin\frac{\pi t}{2}.$$

The exponential notation is justified by the usual power series inside the associative two-dimensional algebra C_i.

The name "curl" is mnemonic: e_i is the quarter-turn direction in the local complex slice. No vector-calculus curl operator is being invoked. Equivalently, this is the quarter-turn bridge of Theorem 7.2 with $J = e_i$ acting by left multiplication on the slice C_i.

Theorem 8.2 (Fractional curl localization)**.** *For all $s, t \in \mathbb{R}$,*

$$\mathrm{Curl}_i(t+s) = \mathrm{Curl}_i(t)\,\mathrm{Curl}_i(s),$$
$$\mathrm{Curl}_i(t)^* = \mathrm{Curl}_i(-t) = \mathrm{Curl}_i(t)^{-1},$$

and

$$N_i(\mathrm{Curl}_i(t)) = 1.$$

Under the local chart $\phi_i : C_i \to \mathbb{C}$,

$$\phi_i(\mathrm{Curl}_i(t)) = \exp\left(i\frac{\pi t}{2}\right).$$

In particular,

$$\mathrm{Curl}_i(0) = 1, \qquad \mathrm{Curl}_i(1) = e_i, \qquad \mathrm{Curl}_i(2) = -1, \qquad \mathrm{Curl}_i(3) = -e_i, \qquad \mathrm{Curl}_i(4) = 1.$$

Proof. By Theorem 2.4, ϕ_i is an isomorphism and sends e_i to the ordinary imaginary unit i. Hence

$$\phi_i(\mathrm{Curl}_i(t)) = \cos\frac{\pi t}{2} + i\sin\frac{\pi t}{2} = \exp\left(i\frac{\pi t}{2}\right).$$

The multiplicative, inverse, and norm statements are the pullbacks of the corresponding identities for complex exponentials. The listed special values follow by substituting $t = 0, 1, 2, 3, 4$. □

Corollary 8.3 (Half-turn periodicity)**.** *For all $t \in \mathbb{R}$,*

$$\mathrm{Curl}_i(t+2) = -\,\mathrm{Curl}_i(t), \qquad \mathrm{Curl}_i(t+4) = \mathrm{Curl}_i(t).$$

The unoriented line

$$L_i(t) = \mathbb{R}\,\mathrm{Curl}_i(t)$$

satisfies

$$L_i(t+2) = L_i(t).$$

Proof. The first two identities follow from Theorem 8.2 and the special values $\mathrm{Curl}_i(2) = -1$, $\mathrm{Curl}_i(4) = 1$. Multiplication by -1 does not change the real line spanned by an element, so $L_i(t+2) = L_i(t)$. □

8.2. **Subunit activation weights.** The language of activation weights must be normalized carefully. A real vector-space dimension is not being changed by a non-integer amount. Instead, a weight of a unit activation is encoded by a fractional quarter-turn inside an already existing ALH slice. Thus the parameter t below measures position along the path from 1 to the oriented generator e_i, not the cardinality or dimension of a vector space.

Definition 8.4 (Normalized activation weight)**.** Let $a_1, \ldots, a_m$ be real numbers with total

$$A = a_1 + \cdots + a_m \neq 0.$$

The normalized activation weight of a_r is

$$t_r = \frac{a_r}{A}.$$

The corresponding curl angle is

$$\theta_r = \frac{\pi}{2} t_r = \frac{a_r \pi}{2(a_1 + \cdots + a_m)}.$$

For two positive weights a, b with $a + b = 1$, this is exactly

$$\theta_a = \frac{a\pi}{2a + 2b} = \frac{a\pi}{2}.$$

Theorem 8.5 (Composition of real activation weights)**.** *Let $u_1, \ldots, u_m \in \mathbb{R}$ and set*

$$U = u_1 + \cdots + u_m.$$

Then

$$\mathrm{Curl}_i(u_1) \cdots \mathrm{Curl}_i(u_m) = \mathrm{Curl}_i(U).$$

Consequently,

$$U = 1 \quad \Longrightarrow \quad \mathrm{Curl}_i(u_1) \cdots \mathrm{Curl}_i(u_m) = e_i,$$

and

$$U = -1 \quad \Longrightarrow \quad \mathrm{Curl}_i(u_1) \cdots \mathrm{Curl}_i(u_m) = -e_i.$$

Equivalently, if $a_1, \ldots, a_m$ have nonzero total A and the intended oriented target is $\varepsilon \in \{+1, -1\}$, set

$$t_r = \varepsilon \frac{a_r}{A}.$$

Then $t_1 + \cdots + t_m = \varepsilon$ and

$$\mathrm{Curl}_i(t_1) \cdots \mathrm{Curl}_i(t_m) = \varepsilon e_i.$$

Proof. By Theorem 8.2, the product equals

$$\mathrm{Curl}_i(u_1 + \cdots + u_m) = \mathrm{Curl}_i(U).$$

The cases $U = 1$ and $U = -1$ use $\mathrm{Curl}_i(1) = e_i$ and $\mathrm{Curl}_i(-1) = -e_i$. The normalized-target statement follows because the chosen t_r sum to ε. □

Remark 8.6 (Why $a + be_i$ is not the right encoding). If $a, b > 0$ and $a + b = 1$, the element $a + be_i$ generally has

$$N_i(a + be_i) = a^2 + b^2 < 1.$$

It is therefore not a pure rotation. A subunit activation share is encoded by

$$\cos \frac{a\pi}{2} + e_i \sin \frac{a\pi}{2},$$

not by the coefficient vector $a + ae_i$ or $a + be_i$.

9. Complex curl activation interface

9.1. **Definition and basic properties.** Real activation parameters give unit-norm rotations. Complex activation parameters add a radial scale. This is the ALH meaning of a complex-valued activation parameter: the real part is an angle parameter, while the imaginary part is a logarithmic scale parameter inside the same two-dimensional real slice.

Definition 9.1 (Complex curl activation)**.** Let $z = x + iy \in \mathbb{C}$. Define

$$\mathrm{Chi}_i(z) = \phi_i^{-1}\left(\exp\left(i\frac{\pi z}{2}\right)\right).$$

Equivalently,

$$\mathrm{Chi}_i(x+iy) = e^{-\pi y/2}\left(\cos\frac{\pi x}{2} + e_i \sin\frac{\pi x}{2}\right).$$

This definition uses the complex chart only to define an element of the real slice C_i. The final element has real coefficients in the basis $\{1, e_i\}$. Thus complex activation is not an extra scalar extension of the ambient algebra; it is the pullback, through the local ALH chart, of the ordinary complex exponential.

Theorem 9.2 (Complex activation theorem)**.** *For all $z, w \in \mathbb{C}$,*

$$\mathrm{Chi}_i(z+w) = \mathrm{Chi}_i(z)\,\mathrm{Chi}_i(w),$$
$$\mathrm{Chi}_i(z)^{-1} = \mathrm{Chi}_i(-z),$$
$$\mathrm{Chi}_i(z)^* = \mathrm{Chi}_i(-\overline{z}),$$

and

$$N_i(\mathrm{Chi}_i(z)) = e^{-\pi \operatorname{Im} z}.$$

Moreover

$$\mathrm{Chi}_i(z+4) = \mathrm{Chi}_i(z), \qquad \mathrm{Chi}_i(z+2) = -\mathrm{Chi}_i(z).$$

Proof. Apply ϕ_i. Since

$$\phi_i(\mathrm{Chi}_i(z)) = e^{i\pi z/2},$$

we get

$$\phi_i(\mathrm{Chi}_i(z+w)) = e^{i\pi(z+w)/2} = e^{i\pi z/2}e^{i\pi w/2} = \phi_i(\mathrm{Chi}_i(z)\,\mathrm{Chi}_i(w)).$$

Injectivity of ϕ_i gives the first identity. The inverse identity follows from $e^{i\pi z/2}e^{-i\pi z/2} = 1$.

For the $*$-identity,

$$\phi_i(\mathrm{Chi}_i(z)^*) = \overline{e^{i\pi z/2}} = e^{-i\pi\overline{z}/2} = e^{i\pi(-\overline{z})/2} = \phi_i(\mathrm{Chi}_i(-\overline{z})).$$

Thus $\mathrm{Chi}_i(z)^* = \mathrm{Chi}_i(-\overline{z})$. For the norm,

$$N_i(\mathrm{Chi}_i(z)) = |e^{i\pi z/2}|^2 = e^{-\pi \operatorname{Im} z}.$$

The periodicity identities follow from $e^{i\pi(z+4)/2} = e^{i\pi z/2}$ and $e^{i\pi(z+2)/2} = -e^{i\pi z/2}$. □

Corollary 9.3 (Real and imaginary roles)**.** *For $z = x + iy$,*

$$\operatorname{Re} z = x$$

controls the curl angle $\pi x/2$, while

$$\operatorname{Im} z = y$$

controls the radial scale $e^{-\pi y/2}$ and the squared norm $e^{-\pi y}$.

Proof. This is the explicit formula in Definition 9.1. □

9.2. **Complex activation sums.**

Theorem 9.4 (Complex activation sum composition)**.** *Let $z_1, \dots, z_m \in \mathbb{C}$. Then*

$$\mathrm{Chi}_i(z_1) \cdots \mathrm{Chi}_i(z_m) = \mathrm{Chi}_i(z_1 + \cdots + z_m).$$

In particular, if

$$z_1 + \cdots + z_m = 1,$$

then

$$\mathrm{Chi}_i(z_1) \cdots \mathrm{Chi}_i(z_m) = e_i.$$

If

$$z_1 + \cdots + z_m = -1,$$

then

$$\mathrm{Chi}_i(z_1) \cdots \mathrm{Chi}_i(z_m) = -e_i.$$

More generally, for $\varepsilon \in \{+1, -1\}$, if

$$z_1 + \cdots + z_m = \varepsilon + i\eta,$$

then

$$\mathrm{Chi}_i(z_1) \cdots \mathrm{Chi}_i(z_m) = \varepsilon e^{-\pi\eta/2} e_i.$$

Proof. The first identity follows by repeated application of Theorem 9.2. The special cases use the explicit values

$$\mathrm{Chi}_i(1) = e_i, \qquad \mathrm{Chi}_i(-1) = -e_i,$$

and, for $\varepsilon \in \{+1, -1\}$,

$$\mathrm{Chi}_i(\varepsilon + i\eta) = \varepsilon e^{-\pi\eta/2} e_i.$$

□

Corollary 9.5 (Negative complex target is equally valid)**.** *Let $z_1, \dots, z_m \in \mathbb{C}$ have total activation*

$$z_1 + \cdots + z_m = -1.$$

Then the ALH activation product is the oriented negative generator

$$\mathrm{Chi}_i(z_1) \cdots \mathrm{Chi}_i(z_m) = -e_i.$$

Equivalently, the condition $z_1 + \cdots + z_m = \varepsilon$ with $\varepsilon \in \{+1, -1\}$ gives εe_i. The case $\varepsilon = -1$ is not exceptional; it is the same local complex line with the opposite curl orientation.

Proof. This is Theorem 9.4 with $\eta = 0$ and $\varepsilon = -1$. □

Remark 9.6 (Cancellation of imaginary activation). The individual activation parameters may have nonzero imaginary parts. For instance, if $z_1 = 1/2 + i\eta$ and $z_2 = -3/2 - i\eta$, then $z_1 + z_2 = -1$ and hence

$$\mathrm{Chi}_i(z_1)\,\mathrm{Chi}_i(z_2) = -e_i.$$

The imaginary activations cancel in the total. If the total were $-1 + i\eta_0$ instead, the product would be $-e^{-\pi\eta_0/2}e_i$, carrying the same negative orientation but with radial scaling.

Remark 9.7 (Complex activation parameter versus dimension). The phrase "complex dimension" is avoided in the formal statements. A vector-space dimension is a natural number or a cardinal. In ALH, $z \in \mathbb{C}$ is a complex activation parameter. Its real part rotates inside the local complex slice; its imaginary part scales the norm. This is well-defined, but it is not a statement that the basis cardinality of a vector space has become complex.

10. Orthogonal coordinate realization

The curl language is not tied to Cartesian coordinates. The essential hypothesis is the existence of an oriented orthonormal two-frame. Once such a frame is available, the ALH quarter-turn is automatic. Every regular orthogonal coordinate chart supplies such frames after the coordinate basis is normalized by its scale factors. Thus Cartesian, polar, spherical, and standard orthogonal toroidal or tokamak-type coordinates require no additional algebraic input. This section is therefore not a new coordinate-dependent theorem; it is an application of the quarter-turn bridge to normalized coordinate frames. If a coordinate system is nonorthogonal, ALH applies after replacing the raw coordinate basis by a local orthonormal frame.

Algebraically, the coordinate system only supplies an orthonormal pair of real directions. Geometrically, the normalization by the scale factors h_a removes the metric weights from the coordinate basis. After normalization, each ordered pair of directions behaves like the standard Euclidean plane with its $\pi/2$ rotation.

Definition 10.1 (Regular orthogonal coordinate frame)**.** Let U be a coordinate patch in a Riemannian manifold or in Euclidean space with coordinates

$$(q^1, \dots, q^n)$$

whose metric is diagonal,

$$ds^2 = \sum_{a=1}^{n} h_a(q)^2 (dq^a)^2, \qquad h_a(q) > 0.$$

The normalized coordinate frame is

$$E_a = \frac{1}{h_a}\frac{\partial}{\partial q^a}.$$

For an ordered pair $a \neq b$, define the oriented coordinate two-plane

$$P_{ab} = \operatorname{span}_{\mathbb{R}}\{E_a, E_b\}$$

and the positive quarter-turn $J_{ab} : P_{ab} \to P_{ab}$ by

$$J_{ab}E_a = E_b, \qquad J_{ab}E_b = -E_a.$$

Theorem 10.2 (Orthogonal-coordinate ALH principle)**.** *On every regular orthogonal coordinate patch and for every ordered pair $a \neq b$, the quarter-turn J_{ab} satisfies*

$$J_{ab}^2 = -I_{P_{ab}}, \qquad J_{ab}^* = -J_{ab}$$

with respect to the Riemannian inner product on P_{ab}. Hence

$$C_{ab} = \{xI_{P_{ab}} + yJ_{ab} : x, y \in \mathbb{R}\}$$

is a local ALH slice and is $*$-isomorphic to $\mathbb{C}$ by

$$xI_{P_{ab}} + yJ_{ab} \longmapsto x + iy.$$

The corresponding curl activation is the coordinate-plane rotation

$$\operatorname{Curl}_{ab}(t) = \cos\frac{\pi t}{2} I_{P_{ab}} + \sin\frac{\pi t}{2} J_{ab}.$$

In particular, subunit, negative, and complex activations defined in Definitions 8.1 and 9.1 apply unchanged in any regular orthogonal coordinate system.

Proof. Because the frame $\{E_a\}$ is orthonormal, the restriction of the metric to P_{ab} has matrix I_2 in the ordered basis (E_a, E_b). In that basis

$$J_{ab} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Therefore $J_{ab}^2 = -I_2$ and $J_{ab}^T = -J_{ab}$, which is exactly $J_{ab}^* = -J_{ab}$. The local $*$-isomorphism statement is exactly the quarter-turn bridge, Theorem 7.2, applied to J_{ab}. The displayed formula for $\operatorname{Curl}_{ab}(t)$ is the exponential $\exp((\pi t/2)J_{ab})$ in this two-plane. □

Remark 10.3 (Locality and coordinate singularities). The theorem is local on the regular part of the coordinate chart. Polar coordinates exclude $r = 0$; spherical coordinates exclude $r = 0$ and the polar axis; toroidal or tokamak-type coordinates exclude the usual coordinate degeneracy sets. These are chart singularities, not ALH obstructions. On every regular patch, the normalized orthogonal frame gives the same local quarter-turn calculus.

10.1. **Standard coordinate systems.**

Example 10.4 (Cartesian coordinates)**.** In Cartesian coordinates on $\mathbb{R}^n$, all scale factors are $h_a = 1$. For the (x, y)-plane, $E_x = \partial_x$, $E_y = \partial_y$, and

$$J_{xy}\partial_x = \partial_y, \qquad J_{xy}\partial_y = -\partial_x.$$

This is the standard positive $\pi/2$ rotation. Thus $+1$ activation gives J_{xy} and -1 activation gives $-J_{xy}$.

Example 10.5 (Polar coordinates)**.** On the punctured plane with polar coordinates (r, θ),

$$ds^2 = dr^2 + r^2 d\theta^2, \qquad E_r = \partial_r, \qquad E_\theta = \frac{1}{r}\partial_\theta.$$

The ALH quarter-turn is

$$J_{r\theta} E_r = E_\theta, \qquad J_{r\theta} E_\theta = -E_r.$$

The factor r is absorbed by normalization; the ALH slice is therefore the same local complex plane as in Cartesian coordinates, expressed in curvilinear coordinates.

Example 10.6 (Spherical coordinates)**.** For spherical coordinates (r, θ, φ) away from $r = 0$ and $\sin\theta = 0$,

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\varphi^2,$$

The normalized frame is

$$E_r = \partial_r, \qquad E_\theta = \frac{1}{r}\partial_\theta, \qquad E_\varphi = \frac{1}{r\sin\theta}\partial_\varphi.$$

Each ordered coordinate plane, for example (E_θ, E_φ), carries a quarter-turn $J_{\theta\varphi}$ with $J_{\theta\varphi}^2 = -I_{P_{\theta\varphi}}$ and $J_{\theta\varphi}^* = -J_{\theta\varphi}$. Hence ALH activation is valid in radial-tangential, meridional, and azimuthal local slices.

Example 10.7 (Orthogonal toroidal and tokamak-type coordinates)**.** In the standard circular toroidal model used as a local tokamak-type coordinate system,

$$R = R_0 + r\cos\theta, \qquad Z = r\sin\theta, \qquad \varphi = \varphi,$$

one has, on the regular region $R_0 + r\cos\theta > 0$,

$$ds^2 = dr^2 + r^2 d\theta^2 + (R_0 + r\cos\theta)^2 d\varphi^2.$$

Thus

$$E_r = \partial_r, \qquad E_\theta = \frac{1}{r}\partial_\theta, \qquad E_\varphi = \frac{1}{R_0 + r\cos\theta}\partial_\varphi.$$

The poloidal-toroidal plane $\mathrm{span}\{E_\theta, E_\varphi\}$ has a canonical oriented quarter-turn once the curl orientation is chosen. Consequently $C_{\theta\varphi} = \mathbb{R} I_{P_{\theta\varphi}} \oplus \mathbb{R} J_{\theta\varphi}$ is an ALH slice. The same conclusion holds for any orthogonal tokamak coordinate model with diagonal metric scale factors. If a plasma coordinate system is nonorthogonal, one must first choose an orthonormal frame, for example by local Gram–Schmidt or a metric frame field; ALH then applies to that frame, not to the raw nonorthogonal coordinate basis.

11. CRE-type lifting invariance

ALH is independent of CRE, but it proves the local hypothesis needed by any CRE-type lifting. Such a lifting is admissible when its chosen imaginary direction is a skew-adjoint complex structure. Once this is true, the lifting is automatically a $*$-isomorphic realization of the local ALH slice. In the terminology of Section 7, every admissible lift is simply a choice of admissible quarter-turn J. The usual real block CRE convention is one concrete display of this mechanism; the local theorem itself also covers complex-slice operator realizations and phase–scale activation realizations.

Thus the word "lifting" is used in a local algebraic sense. It means that the abstract slice $\mathbb{C}$ or C_i is represented by operators generated by I_H and J. The target may be a real block model, a complex-slice operator model, or an activation model inside the same two-dimensional slice. What matters for ALH is the pair of equations $J^2 = -I_H$ and $J^* = -J$.

The same local equations also explain the null-gauge cancellation that appears in CRE-type channel decompositions. If a compensating channel D is defined on a common J-invariant domain so that JD and $J(JD)$ are defined, then

$$D + J(JD) = D + J^2 D = D - D = 0.$$

The displayed identity is an algebraic identity on that common invariant domain. Stronger compatibility assumptions, such as commutation with J on the domain, may be imposed in applications but are not needed for this elementary cancellation.

Definition 11.1 (Admissible CRE-type scalar lift)**.** Let H be a nonzero real Hilbert space and let $J \in B(H)$ satisfy

$$J^2 = -I_H, \qquad J^* = -J,$$

where I_H is the identity on H. The scalar CRE-type lift determined by J is

$$\rho_J : \mathbb{C} \to B(H), \qquad \rho_J(a + bi) = aI_H + bJ.$$

Proposition 11.2 (Scalar CRE-type lift is a $*$-monomorphism)**.** *For every admissible J, the map ρ_J is a unital real $*$-algebra monomorphism. Its image*

$$\rho_J(\mathbb{C}) = \{aI_H + bJ : a, b \in \mathbb{R}\}$$

is a local ALH slice in $B(H)$.

Proof. By Theorem 7.2, the algebra $C_J = \{aI_H + bJ : a, b \in \mathbb{R}\}$ is a local ALH slice and the map $a + bi \mapsto aI_H + bJ$ is a real $*$-isomorphism onto that slice. This proves real linearity, preservation of the unit, multiplication, the $*$-operation, and injectivity. The image is exactly $\rho_J(\mathbb{C}) = C_J$. □

Definition 11.3 (Matrix CRE-type lift along J)**.** For $A = X + iY \in M_n(\mathbb{C})$ with $X, Y \in M_n(\mathbb{R})$, define

$$\rho_J^{(n)}(A) = X \otimes I_H + Y \otimes J$$

on $\mathbb{R}^n \otimes H$. When $H = \mathbb{R}^2$ and

$$J_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

this is the usual real block display

$$X + iY \longmapsto \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix}.$$

Using $-J_0$ gives the conjugation-normalized block convention.

Theorem 11.4 (All admissible CRE-type liftings are local $*$-isomorphic)**.** *Let J be admissible. Then*

$$\rho_J^{(n)} : M_n(\mathbb{C}) \to B(\mathbb{R}^n \otimes H)$$

is a unital real $*$-algebra monomorphism:

$$\rho_J^{(n)}(AB) = \rho_J^{(n)}(A)\rho_J^{(n)}(B), \qquad \rho_J^{(n)}(A^*) = \rho_J^{(n)}(A)^*.$$

If $J' = UJU^*$ for an orthogonal operator U, then

$$\rho_{J'}^{(n)}(A) = (I_n \otimes U)\rho_J^{(n)}(A)(I_n \otimes U)^*.$$

Thus the two CRE-type liftings are $*$-isomorphic by conjugation. If $J' = -J$, then

$$\rho_{-J}^{(n)}(A) = \rho_J^{(n)}(\overline{A}),$$

so reversing the curl orientation gives the conjugate local complex chart, not a failure of the CRE lift.

Proof. The scalar statement is the quarter-turn bridge. The matrix statement is its entrywise amplification, but we spell out the calculation to fix the block convention.

Let $A = X + iY$ and $B = U + iV$, with $X, Y, U, V \in M_n(\mathbb{R})$. Then

$$\begin{aligned} \rho_J^{(n)}(A)\rho_J^{(n)}(B) &= (X \otimes I_H + Y \otimes J)(U \otimes I_H + V \otimes J) \\ &= (XU - YV) \otimes I_H + (XV + YU) \otimes J \\ &= \rho_J^{(n)}(AB), \end{aligned}$$

using $J^2 = -I_H$. For adjoints,

$$\rho_J^{(n)}(A)^* = X^T \otimes I_H + Y^T \otimes J^* = X^T \otimes I_H - Y^T \otimes J = \rho_J^{(n)}(A^*).$$

Injectivity follows because I_H and J are $\mathbb{R}$-linearly independent and the scalar lift $a + bi \mapsto aI_H + bJ$ is injective. Indeed, if $X \otimes I_H + Y \otimes J = 0$, then each matrix coefficient gives a relation $x_{rs}I_H + y_{rs}J = 0$, hence $x_{rs} = y_{rs} = 0$ for all r, s.

If $J' = UJU^*$, then

$$X \otimes I_H + Y \otimes J' = (I_n \otimes U)(X \otimes I_H + Y \otimes J)(I_n \otimes U)^*,$$

which proves conjugacy. Finally,

$$\rho_{-J}^{(n)}(X + iY) = X \otimes I_H - Y \otimes J = \rho_J^{(n)}(X - iY) = \rho_J^{(n)}(\overline{A}).$$

□

Corollary 11.5 (CRE-type lifting is independent of the chosen orthogonal frame)**.** *Suppose a CRE-type construction chooses its imaginary direction from a Cartesian, polar, spherical, orthogonal toroidal, tokamak-type, signed-XOR, activation, or other ALH source. If the chosen lift produces an operator J with*

$$J^2 = -I_H, \qquad J^* = -J,$$

then the resulting representation is a $*$-isomorphic lift of the local complex algebra. Changing the orthogonal frame or the admissible lift changes the representation by orthogonal conjugacy, while reversing orientation changes the chart by complex conjugation.

Proof. The coordinate sources listed above provide admissible J's by Theorem 10.2 or by the local ALH theorem. The conclusion is Theorem 11.4. □

11.1. **CRE-type liftings as consequences of ALH.** The preceding results show that CRE does not add a new local algebraic assumption to ALH. Once a lift supplies a skew-adjoint complex structure J, the quarter-turn bridge forces

$$\mathbb{C} \cong_{\mathbb{R},*} C_J = \{aI_H + bJ : a, b \in \mathbb{R}\},$$

and the matrix-level CRE map is one entrywise amplification of this local slice isomorphism.

This is the precise sense in which ALH strengthens the foundation of CRE-type liftings: the existence of a local complex chart is no longer treated as a representation convention, but as a theorem following from $J^2 = -I_H$ and $J^* = -J$. Conversely, such liftings broaden the scope of ALH by giving every admissible quarter-turn a concrete operator representation on a Hilbert block. Thus the relation is bidirectional at the local algebraic level:

$$\text{ALH local slice} \iff \text{admissible quarter-turn} \implies \text{CRE-type } *\text{-representation}.$$

The first equivalence is local: a generator e_i acts as a quarter-turn on its two-dimensional slice, and any admissible quarter-turn J generates the same algebraic copy of $\mathbb{C}$. The final implication is representational: a CRE-type lifting realizes that local copy of $\mathbb{C}$ as operators or block matrices.

Only this local slice-representation aspect is used here. Broader operator-framework applications of CRE may impose additional hypotheses such as positivity or spectral conditions, but those hypotheses are external to ALH and play no role in the local $*$-isomorphism theorem.

Remark 11.6 (Scope of the CRE statement)*.* The theorem proves the local algebraic fact needed by CRE-type liftings: admissible liftings preserve multiplication and $*$ exactly. Positivity, conditioning, spectral comparisons, and arithmetic constructions are additional analytic or arithmetic statements. They are not consequences of ALH alone and are not used in the proof of the local slice theorem.

12. Examples

12.1. **Classical complex numbers.** Let $A = \mathbb{C}$ regarded as a real $*$-algebra and let $e = i$. Then

$$C_e = \mathbb{R}1 \oplus \mathbb{R}i = \mathbb{C},$$

and ALH is the identity theorem. The curl map is the usual exponential

$$\mathrm{Curl}_e(t) = e^{i\pi t/2}.$$

12.2. **Quaternions.** Let $A = \mathbb{H}$ with generators $\mathbf{i}, \mathbf{j}, \mathbf{k}$. Each of

$$\mathbb{R}1 \oplus \mathbb{R}\mathbf{i}, \quad \mathbb{R}1 \oplus \mathbb{R}\mathbf{j}, \quad \mathbb{R}1 \oplus \mathbb{R}\mathbf{k}$$

is an ALH complex slice. The pair $(\mathbf{i}, \mathbf{j})$ generates the quaternionic compatible slice. Orientation reversal sends $\mathbf{i}$ to $-\mathbf{i}$, giving the same complex plane with conjugated chart.

12.3. **Sorted Clifford algebras.** In $Cl_{0,n}(\mathbb{R})$, every generator e_i satisfies $e_i^2 = -1$ and Clifford conjugation sends e_i to $-e_i$. Hence every generator gives an ALH slice. Every pair of distinct generators gives a quaternionic slice. The full algebra is not generally a division algebra; ALH makes only the local statement.

12.4. **Complex-labelled generators.** Let $I = \mathbb{C}$ with lexicographic order, and construct $A_{\mathrm{sort}}(I)$. For every label $z \in \mathbb{C}$ there is a generator e_z with

$$e_z^2 = -1.$$

Then

$$C_z = \mathbb{R}1 \oplus \mathbb{R}e_z \cong \mathbb{C}.$$

The label z may also carry phase metadata if desired, but the local ALH theorem does not require using z as a scalar. If an oriented phase lift declares $e_{\bar{z}} = -e_z$, then $C_{\bar{z}} = C_z$ and the charts differ by conjugation.

13. Conclusion

ALH is best understood as a functorial local $*$-isomorphism theorem. Its invariant core is

$$\forall i \in I, \qquad \mathbb{R}1 \oplus \mathbb{R}e_i \cong_{\mathbb{R},*} \mathbb{C}.$$

This statement survives arbitrary reindexing, arbitrary label sets, orientation reversal, finite-support enlargement of the ambient algebra, and passage to regular orthogonal coordinate frames. In a normalized orthogonal frame, the positive quarter-turn J always satisfies $J^2 = -I_P$ and $J^* = -J$ on the chosen oriented two-plane P, so Cartesian, polar, spherical, and orthogonal toroidal or tokamak-type coordinates all carry the same local ALH calculus.

The real and complex activation maps are

$$\mathrm{Curl}_i(t) = \cos\frac{\pi t}{2} + e_i \sin\frac{\pi t}{2},$$

$$\mathrm{Chi}_i(z) = e^{-\pi \operatorname{Im} z/2}\left(\cos\frac{\pi \operatorname{Re} z}{2} + e_i \sin\frac{\pi \operatorname{Re} z}{2}\right).$$

Real parameters are unit curl rotations. Complex parameters are invertible phase-scale transformations inside the same local complex slice. A total activation $+1$ gives e_i; a total activation -1 gives $-e_i$. The negative case is not singular; it is the reversed orientation of the same imaginary axis.

The signed-XOR backend shows how these local slices can be embedded in a table-free finite-support algebra. In the sorted sector, the product is computed by XOR support and a parity sign; in more general twists, the associator is explicitly measured by the twist. This makes the distinction sharp: local ALH arithmetic is universal, while global associativity, division, and norm composition require additional finite-slice hypotheses.

Finally, CRE-type lifting is already settled at the ALH level by the quarter-turn bridge. Any admissible lift with $J^2 = -I_H$ and $J^* = -J$ gives a $*$-monomorphic representation of the local complex algebra, and any two conjugate choices of J give conjugate representations. Reversing the curl orientation only replaces the chart by complex conjugation. This makes the ALH foundation stronger, because admissibility is reduced to the same two local equations that define a quarter-turn; it also widens the operational scope of ALH, because every such quarter-turn yields a concrete operator representation, including but not limited to real block CRE models. Null-gauge cancellations such as $D + J(JD) = 0$ are then seen as local quarter-turn cancellations on any common J-invariant domain where the products JD and $J(JD)$ are defined.

Thus ALH is not an escape from Hurwitz's theorem. It is the precise local calculus that remains available in every dimension, over every index system, and in every regular orthogonal coordinate realization.

References

1. Helena Albuquerque and Shahn Majid, *Quasialgebra structure of the octonions*, Journal of algebra **220** (1999), no. 1, 188–224.
2. Leonard E. Dickson, *On quaternions and their generalization and the history of the eight square theorem*, Annals of Mathematics **20** (1919), no. 3, 155–171.
3. Adolf Hurwitz, *über die Komposition der quadratischen Formen*, Tech. report, Springer, 1963.
4. Junhu, Park, *AXIOM: Algebraic eXtension by Index–Ordered Multiplication*, Master's thesis, Seoul National University, 2026.
5. ———, *Signed-XOR Structural Constants and Table-Free Multiplication in Clifford and Cayley-Dickson Algebras*, (2026).
6. Pertti Lounesto, *Clifford Algebras and Spinors*, Clifford Algebras and Their Applications in Mathematical Physics (J. S. R. Chisholm and A. K. Common, eds.), Springer Netherlands, Dordrecht, 1986, pp. 25–37.
7. Ian R. Porteous, *Clifford algebras and the classical groups*, no. 50, Cambridge University Press, 1995.

8. R. D. Schafer, *An Introduction to Nonassociative Algebras*, American Mathematical Society **61** (1955), 469–484.
9. Tonny A. Springer and Ferdinand D. Veldkamp, *Composition Algebras*, pp. 1–23, Springer Berlin Heidelberg, Berlin, Heidelberg, 2000.

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