HOMOGENEOUS ARITHMETIC LOCAL HURWITZ QUARTER-TURN ATLASES IN SIGNED-XOR, CLIFFORD-LIKE, AND CAYLEY–DICKSON ALGEBRAS

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Abstract. Arithmetic Local Hurwitz (ALH) is usually visible first as a local statement: an element e satisfying $e^2 = -1$ and $e^* = -e$ spans a canonical real $*$-algebraic copy $\mathbb{R}\,1 \oplus \mathbb{R}e \cong \mathbb{C}$. This paper develops the corresponding global finite-support structure. The main object is not a global division algebra or a global Hurwitz composition law, but a homogeneous atlas of quarter-turn two-planes carried by signed-XOR and Clifford-like basis systems.

Let V be an elementary abelian two-group and let $\mathbb{R}_\theta[V]$ be a signed-XOR algebra with basis $\{e_\alpha : \alpha \in V\}$ and product

$$e_\alpha e_\beta = \theta(\alpha,\beta)e_{\alpha+\beta}, \qquad \theta(\alpha,\beta) \in \{\pm 1\}.$$

For a fixed direction $\delta \in V$, right multiplication $R_\delta(e_\alpha) = e_\alpha e_\delta$ is a global ALH quarter-turn precisely when

$$\theta(\alpha,\delta)\theta(\alpha+\delta,\delta) = -1 \qquad (\alpha \in V).$$

In that case $R_\delta^2 = -I$ and $R_\delta^\dagger = -R_\delta$ with respect to the orthonormal basis inner product. The support toggle $\alpha \mapsto \alpha + \delta$ decomposes the whole space into homogeneous ALH two-planes

$$\mathbb{R}_\theta[V] = \bigoplus_{[\alpha]\in V/\langle\delta\rangle} \operatorname{span}_{\mathbb{R}}\{e_\alpha, e_{\alpha+\delta}\},$$

where R_δ is the standard quarter-turn after an orientation choice on each orbit. Thus, once a direction δ is right-ALH-admissible for the twist θ, the support-orbit decomposition depends only on $V/\langle\delta\rangle$. The twist records the local orientation signs and associator data.

The Cayley–Dickson tower is then recovered as a distinguished layer-flow case. For a doubling

$$A_{n+1} = A_n \oplus A_n\varepsilon_n,$$

right multiplication by the newly adjoined unit satisfies

$$J_n = R_{\varepsilon_n}, \qquad J_n^2 = -I, \quad J_n^\dagger = -J_n,$$

without using associativity, alternativity, division, or norm multiplicativity. Moreover

$$J_n(A_n) = A_n\varepsilon_n, \qquad J_n(A_n\varepsilon_n) = -A_n,$$

so the upward layer transition is the positive quarter-turn and the inverse downward transition is the negative quarter-turn. In basis form,

$$e_r \mapsto e_{r+2^n}, \qquad e_{r+2^n} \mapsto -e_r.$$

The quaternionic rule $1 \mapsto j$ and $i \mapsto k$ is the first nontrivial instance; the same theorem applies to octonions, sedenions, and all higher Cayley–Dickson stages.

Contents

1. Introduction 3
Main results 4

Scope 5
2. Local ALH slices 5
3. Quarter-turn operators and axis-relative charts 7
4. Signed-XOR algebras 8
5. The homogeneous quarter-turn atlas 10
6. Clifford-like sectors 12
7. Cayley–Dickson layer-flow theorem 14
8. Quaternions, octonions, sedenions, and beyond 17
9. Relation between the signed-XOR and Cayley–Dickson pictures 19
10. Functoriality and reindexing 22
11. Interpretation: dimension-flow, not fractional dimension 23
12. Conclusion 23
References 24

1. Introduction

Hurwitz-type theorems constrain global finite-dimensional normed division algebras [7, 11, 14]. They do not prevent a larger algebraic system from carrying many local complex structures. Arithmetic Local Hurwitz (ALH) isolates this local residue: whenever a real unital $*$-algebraic system contains an element e with

$$e^2 = -1, \qquad e^* = -e, \tag{1.1}$$

then the two-dimensional real plane $\mathbb{R}\,1 \oplus \mathbb{R}e$ is a copy of $\mathbb{C}$. This local fact is elementary, but it becomes more structural when one asks how such two-planes are distributed through finite-support algebras.

The purpose of the present paper is to make that distribution precise. The main claim is that ALH is not merely a list of isolated complex slices. In signed-XOR and Clifford-like basis systems, admissible imaginary directions determine a homogeneous quarter-turn atlas on the whole basis space, in the spirit of signed structural-constant descriptions of Clifford and Cayley–Dickson multiplication [8]. In Cayley–Dickson doublings this atlas is exactly the layer transition from the lower component to the newly adjoined component.

The motivating example is the quaternionic slice [4, 5, 9]

$$\mathbb{H} = \operatorname{span}_{\mathbb{R}}\{1, i, j, k\}, \qquad ij = k.$$

If the original local slice is $\operatorname{span}_{\mathbb{R}}\{1, i\}$ and the new transverse unit is j, then right multiplication by j gives

$$1 \longmapsto j, \qquad i \longmapsto k.$$

This is not ordinary multiplication by the old imaginary unit i. It is a transverse dimension-flow operator

$$R_j(x) = xj.$$

It satisfies $R_j^2 = -I$ and $R_j^{-1} = -R_j$. Hence the upward transition is a positive quarter-turn, while the downward transition is the negative quarter-turn. The same pattern persists in the octonions [1, 2, 14],

$$\mathbb{O} = \mathbb{H} \oplus \mathbb{H}\ell,$$

and in the sedenions and higher Cayley–Dickson stages [12, 13],

$$\mathbb{S} = \mathbb{O} \oplus \mathbb{O}s,$$

even though global associativity, alternativity, division, and norm composition have already degraded or failed. The layer-flow operator survives because it is a linear quarter-turn, not a global composition theorem.

Main results. The paper proves five linked structure results.

First, the elementary ALH slice theorem is recorded in a form independent of global associativity. The equations $e^2 = -1$ and $e^* = -e$ force $\mathbb{R}\,1 \oplus \mathbb{R}e$ to be a local real $*$-algebraic copy of $\mathbb{C}$.

Second, axis-relative charts are formalized. In a real plane with an admissible quarter-turn J, no axis is absolute. If u is a unit axis, then (u, Ju) is a local real-imaginary frame. If u is the old imaginary axis, then the old real axis is the negative imaginary axis in the new frame. This explains the rule that changing the reference axis changes which direction is read as the quarter-turn.

Third, the paper proves the signed-XOR atlas theorem. Let V be an elementary abelian two-group and let

$$e_\alpha e_\beta = \theta(\alpha, \beta) e_{\alpha+\beta}$$

be a signed-XOR product. A direction δ is right-ALH-admissible if

$$\theta(\alpha, \delta)\theta(\alpha + \delta, \delta) = -1 \qquad (\alpha \in V).$$

Then right multiplication R_δ is a skew-adjoint complex structure on the whole real basis space. The orbits of the support toggle $\alpha \mapsto \alpha + \delta$ give the homogeneous quarter-turn atlas

$$\mathbb{R}_\theta[V] = \bigoplus_{[\alpha] \in V/\langle\delta\rangle} \operatorname{span}_{\mathbb{R}}\{e_\alpha, e_{\alpha+\delta}\}.$$

This is the central theorem of the paper.

Fourth, Cayley–Dickson doubling is shown to be a distinguished layerwise instance of this atlas. For

$$A_{n+1} = A_n \oplus A_n \varepsilon_n,$$

the operator $J_n = R_{\varepsilon_n}$ satisfies

$$J_n(a,b) = (-b,a), \qquad J_n^2 = -I, \qquad J_n^\dagger = -J_n.$$

Consequently

$$A_{n+1} = \bigoplus_{r=0}^{2^n-1} \operatorname{span}_{\mathbb{R}}\{e_r, e_{r+2^n}\},$$

and on each summand

$$e_r \mapsto e_{r+2^n}, \qquad e_{r+2^n} \mapsto -e_r.$$

Fifth, the previous results combine into a unified Hurwitz–Clifford–Cayley–Dickson principle. The local Hurwitz slice theorem provides the fundamental ALH quarter-turn. The signed-XOR atlas theorem globalizes that quarter-turn over finite-support basis systems. Clifford sectors arise as associative signed-XOR realizations, while Cayley–Dickson layer flow appears as a distinguished admissible support-toggle direction. Thus Hurwitz local complex slices, Clifford-like basis systems, and the entire Cayley–Dickson tower are governed by the same homogeneous ALH quarter-turn atlas mechanism.

Scope. The results are local, linear, and finite-support. They do not assert that a high-dimensional Cayley–Dickson algebra is a division algebra [5, 13]. They do not assert that a quadratic norm is multiplicative after the classical Hurwitz range. They do not assert global associativity for signed-XOR twists. The claim is instead that an intrinsic homogeneous quarter-turn atlas remains available even when the global multiplicative properties fail.

2. Local ALH slices

Definition 2.1 (Real unital $*$-algebraic system)**.** A real unital $*$-algebraic system is a real unital algebraic system A, not necessarily globally associative, equipped with an $\mathbb{R}$-linear involution $x \mapsto x^*$ such that $1^* = 1$, $(x^*)^* = x$, and

$$(xy)^* = y^* x^*$$

for all products under consideration. Whenever global associativity is needed, it is stated explicitly.

Definition 2.2 (ALH generator and local slice)**.** An element $e \in A$ is an *ALH imaginary generator* if

$$e^2 = -1, \qquad e^* = -e.$$

Its local ALH slice is

$$C_e = \mathbb{R}\,1 \oplus \mathbb{R}e.$$

For $x = a + be \in C_e$ define

$$\overline{x} = x^* = a - be, \qquad N_e(x) = x^* x = a^2 + b^2.$$

Theorem 2.3 (Local ALH complex slice)**.** *Let A be a real unital $*$-algebraic system and let $e \in A$ satisfy $e^2 = -1$ and $e^* = -e$. Then $C_e = \mathbb{R}\,1 \oplus \mathbb{R}e$ is a two-dimensional real unital $*$-subalgebraic slice with multiplication*

$$(a + be)(c + de) = (ac - bd) + (ad + bc)e.$$

The map

$$\phi_e : C_e \longrightarrow \mathbb{C}, \qquad \phi_e(a + be) = a + bi,$$

is a unital real $*$-algebra isomorphism.

Proof. Bilinearity and $e^2 = -1$ give the displayed multiplication, so C_e is closed under products and contains 1. The displayed product is exactly the complex product under $a + be \mapsto a + bi$. Also

$$\phi_e((a + be)^*) = \phi_e(a - be) = a - bi = \overline{\phi_e(a + be)}.$$

Thus ϕ_e preserves multiplication, the unit, and the involution. It is visibly real-linear and bijective.

□

Corollary 2.4 (Local inverse and norm composition)**.** *If $x = a + be \in C_e$ is nonzero, then*

$$x^{-1} = \frac{x^*}{N_e(x)} = \frac{a - be}{a^2 + b^2}.$$

For all $x, y \in C_e$,

$$N_e(xy) = N_e(x)N_e(y).$$

Proof. Both identities are pullbacks of the corresponding identities in $\mathbb{C}$ under Theorem 2.3. Directly,

$$(a + be)(a - be) = a^2 + b^2.$$

□

Remark 2.5 (Locality). The theorem uses only the two equations $e^2 = -1$ and $e^* = -e$. It does not use a multiplication table for all of A, an ordering of all generators, global associativity, or a global norm-composition identity.

3. Quarter-turn operators and axis-relative charts

The local ALH slice can be described without privileging the symbols 1 and i. The coordinate-free datum is a real quarter-turn.

Definition 3.1 (Admissible quarter-turn)**.** Let E be a real inner-product space. A linear operator $J : E \to E$ is an *admissible quarter-turn* if

$$J^2 = -I_E, \qquad J^\dagger = -J,$$

where $J^\dagger$ is the adjoint with respect to the inner product.

Theorem 3.2 (Operator ALH bridge)**.** *If J is an admissible quarter-turn on E, then*

$$C_J = \{aI_E + bJ : a, b \in \mathbb{R}\} \subseteq \mathrm{End}_{\mathbb{R}}(E)$$

is a real unital $*$-subalgebra and

$$aI_E + bJ \longmapsto a + bi$$

is a unital real $*$-algebra isomorphism $C_J \cong \mathbb{C}$.

Proof. The product is

$$(aI + bJ)(cI + dJ) = (ac - bd)I + (ad + bc)J$$

because $J^2 = -I$. The adjoint is

$$(aI + bJ)^\dagger = aI - bJ$$

because $J^\dagger = -J$. This is exactly Theorem 2.3 in the operator algebra $\mathrm{End}_{\mathbb{R}}(E)$.

□

Definition 3.3 (Axis-relative chart)**.** Let J be an admissible quarter-turn on a real two-plane P. For a unit vector $u \in P$, the ordered pair

$$(u, Ju)$$

is the axis-relative ALH chart based at u. The associated coordinate map is

$$\psi_u : \mathbb{C} \longrightarrow P, \qquad \psi_u(a + bi) = au + bJu.$$

Proposition 3.4 (Axis relativity)**.** *Let J be an admissible quarter-turn on an oriented Euclidean two-plane P. If u is a unit vector and $v = Ju$, then the chart based at v reads the old real axis u as the negative imaginary axis:*

$$u = -Jv.$$

Equivalently, if u is declared to have angle 0, then v has angle $\pi/2$; if v is declared to have angle 0, then u has angle $-\pi/2$.

Proof. Since $v = Ju$, applying J gives

$$Jv = J^2u = -u.$$

Thus $u = -Jv$. This is exactly the statement that changing the reference axis from u to v rotates the old real axis by $-\pi/2$ in the new chart. □

Example 3.5 (The old imaginary axis as a new real axis)**.** In the standard complex plane, let $J(z) = zi$. With old chart $(1, i)$, the vector i is the positive imaginary axis. If $u = i$ is chosen as the new real axis, then

$$Ju = i^2 = -1.$$

Hence the old real axis satisfies

$$1 = -Ju,$$

so it is the negative imaginary axis in the new chart. This is the precise axis-relative form of the statement that the old 1 becomes a $-\pi/2$ turn when the old i is used as the reference axis.

4. Signed-XOR algebras

We now pass from a single two-plane to a finite-support basis system.

Definition 4.1 (Elementary two-group)**.** An elementary abelian two-group is a vector space V over $\mathbb{F}_2$. We write its operation additively. For an arbitrary index set I, the finite-support model is

$$\mathrm{Fin}(I) = \{J \subseteq I : |J| < \infty\}$$

with addition given by symmetric difference.

Definition 4.2 (Signed-XOR algebra)**.** Let V be an elementary abelian two-group. A *signed-XOR algebra* over V is a real vector space

$$A = \mathbb{R}_\theta[V] = \bigoplus_{\alpha \in V} \mathbb{R} e_\alpha$$

with product determined on basis elements by

$$e_\alpha e_\beta = \theta(\alpha, \beta) e_{\alpha+\beta}, \qquad \theta(\alpha, \beta) \in \{\pm 1\},$$

where θ is normalized:

$$\theta(0, \alpha) = \theta(\alpha, 0) = 1.$$

The basis is equipped with the Euclidean inner product

$$\langle e_\alpha, e_\beta \rangle = \delta_{\alpha\beta}.$$

The product need not be associative. Its associator is measured by a sign, as in twisted group-algebra and nonassociative structural-constant descriptions [1, 3, 8].

Proposition 4.3 (Associator sign)**.** *In* $\mathbb{R}_\theta[V]$*,*

$$(e_\alpha e_\beta)e_\gamma = \omega_\theta(\alpha,\beta,\gamma)e_\alpha(e_\beta e_\gamma),$$

where

$$\omega_\theta(\alpha,\beta,\gamma) = \frac{\theta(\alpha,\beta)\theta(\alpha+\beta,\gamma)}{\theta(\beta,\gamma)\theta(\alpha,\beta+\gamma)}.$$

The algebra is associative if and only if $\omega_\theta \equiv 1$*.*

Proof. Both parenthesizations are scalar multiples of $e_{\alpha+\beta+\gamma}$. The left coefficient is $\theta(\alpha,\beta)\theta(\alpha+\beta,\gamma)$ and the right coefficient is $\theta(\beta,\gamma)\theta(\alpha,\beta+\gamma)$. Comparing them gives the formula. Associativity is exactly equality for all triples.

□

Definition 4.4 (Right multiplication by a direction)**.** For $\delta \in V$, define the right basis-direction operator

$$R_\delta : A \to A, \qquad R_\delta(x) = xe_\delta.$$

On basis elements,

$$R_\delta(e_\alpha) = \theta(\alpha,\delta)e_{\alpha+\delta}.$$

Definition 4.5 (Right-ALH-admissible direction)**.** A direction $\delta \in V$ is *right-ALH-admissible* for θ if

$$\theta(\alpha,\delta)\theta(\alpha+\delta,\delta) = -1 \qquad \text{for all } \alpha \in V. \tag{4.1}$$

This condition is the signed-XOR form of the quarter-turn equation $R_\delta^2 = -I$. It also gives skew-adjointness automatically.

Theorem 4.6 (Signed-XOR quarter-turn criterion)**.** *Let* $A = \mathbb{R}_\theta[V]$ *and let* $\delta \in V$*. If* $\delta = 0$*, none of the conditions below holds because* $\theta(\alpha,0)\theta(\alpha,0) = 1$ *for all* α*. Hence assume* $\delta \neq 0$*. The following are equivalent:*

(i) δ *is right-ALH-admissible, i.e.* (4.1) *holds.*
(ii) $R_\delta^2 = -I_A$*.*
(iii) $R_\delta^\dagger = -R_\delta$ *and* R_δ *is an orthogonal complex structure.*

Consequently, if δ *is right-ALH-admissible, then*

$$\mathbb{R}I_A \oplus \mathbb{R}R_\delta \cong \mathbb{C}$$

as a real $*$*-algebra inside* $\mathrm{End}_{\mathbb{R}}(A)$*.*

Proof. For a basis vector e_α,

$$R_\delta^2(e_\alpha) = \theta(\alpha,\delta)\theta(\alpha+\delta,\delta)e_\alpha.$$

Thus $R_\delta^2 = -I_A$ if and only if (4.1) holds.

Now restrict R_δ to the two-dimensional support orbit spanned by e_α and $e_{\alpha+\delta}$. Write

$$\sigma_\alpha = \theta(\alpha,\delta), \qquad \tau_\alpha = \theta(\alpha+\delta,\delta).$$

Relative to the ordered basis $(e_\alpha, e_{\alpha+\delta})$, the matrix of R_δ is

$$\begin{pmatrix} 0 & \tau_\alpha \\ \sigma_\alpha & 0 \end{pmatrix}.$$

This matrix squares to $\sigma_\alpha\tau_\alpha I_2$. It is skew-symmetric exactly when $\tau_\alpha = -\sigma_\alpha$, which is equivalent to $\sigma_\alpha\tau_\alpha = -1$ since both signs are ± 1. Hence the paired condition is equivalent to both $R_\delta^2 = -I$ and $R_\delta^\dagger = -R_\delta$. Orthogonality follows because R_δ is a signed permutation of the orthonormal basis. The final statement is Theorem 3.2.

□

5. The homogeneous quarter-turn atlas

The previous criterion gives a global decomposition into two-dimensional ALH planes. This is the main structural object of the paper.

Definition 5.1 (Support orbits)**.** Let $\delta \in V$ be nonzero. The cyclic subgroup generated by δ is

$$\langle\delta\rangle = \{0, \delta\}.$$

The support-toggle orbits are the cosets

$$[\alpha] = \alpha + \langle\delta\rangle = \{\alpha, \alpha+\delta\} \in V/\langle\delta\rangle.$$

Theorem 5.2 (Homogeneous ALH quarter-turn atlas)**.** *Let $A = \mathbb{R}_\theta[V]$ and let $\delta \in V$ be right-ALH-admissible. Then*

$$A = \bigoplus_{[\alpha]\in V/\langle\delta\rangle} P_{[\alpha]}, \qquad P_{[\alpha]} = \operatorname{span}_{\mathbb{R}}\{e_\alpha, e_{\alpha+\delta}\}, \tag{5.1}$$

where the summands are mutually orthogonal two-planes. Each $P_{[\alpha]}$ is R_δ-invariant. If

$$f_\alpha = \theta(\alpha,\delta)e_{\alpha+\delta},$$

then (e_α, f_α) is an oriented orthonormal frame of $P_{[\alpha]}$ and

$$R_\delta(e_\alpha) = f_\alpha, \qquad R_\delta(f_\alpha) = -e_\alpha.$$

Thus every support orbit carries the same ALH quarter-turn model.

Proof. The decomposition (5.1) is the decomposition of the orthonormal basis into disjoint two-element orbits under $\alpha \mapsto \alpha + \delta$. The formula

$$R_\delta(e_\alpha) = \theta(\alpha, \delta) e_{\alpha+\delta} = f_\alpha$$

shows invariance. Moreover,

$$R_\delta(f_\alpha) = \theta(\alpha, \delta) R_\delta(e_{\alpha+\delta}) = \theta(\alpha, \delta)\theta(\alpha + \delta, \delta) e_\alpha = -e_\alpha$$

by right-ALH-admissibility. Hence R_δ is the standard quarter-turn on every $P_{[\alpha]}$ in the frame (e_α, f_α).

□

Remark 5.3 (Quarter-turn planes versus algebra slices). The invariant two-planes

$$P_{[\alpha]} = \mathrm{span}_{\mathbb{R}}\{e_\alpha, e_{\alpha+\delta}\}$$

should not in general be interpreted as unital subalgebras. Except in special cases, they do not contain the identity element e_0.

Their role is linear and operator-theoretic: each is an R_δ-invariant real two-plane carrying the standard quarter-turn structure.

The associated unital ALH slice is instead

$$\mathbb{R} I_A \oplus \mathbb{R} R_\delta \subseteq \mathrm{End}_{\mathbb{R}}(A).$$

Corollary 5.4 (Upward and downward orientation)**.** *Under the hypotheses of Theorem 5.2,*

$$R_\delta^{-1} = -R_\delta.$$

Thus the positive support-toggle flow is the $+\pi/2$ turn on every atlas plane. The inverse support-toggle flow is the corresponding $-\pi/2$ turn.

Proof. Since $R_\delta^2 = -I$, multiplying by $-R_\delta$ gives

$$R_\delta(-R_\delta) = (-R_\delta) R_\delta = I.$$

□

Corollary 5.5 (Support orbit decomposition after admissibility)**.** *For fixed V and nonzero δ, once δ is right-ALH-admissible for the twist θ, the underlying support-level orbit decomposition is determined only by the quotient $V/\langle\delta\rangle$. The twist θ affects admissibility, the signs used to orient the frames (e_α, f_α), and the associator data. It does not affect the orbit decomposition*

$$V = \bigsqcup_{[\alpha] \in V/\langle\delta\rangle} \{\alpha, \alpha + \delta\}.$$

Proof. The orbit decomposition uses only addition in V. The twist appears only in the sign

$$f_\alpha = \theta(\alpha, \delta) e_{\alpha+\delta}.$$

By Theorem 5.2, that sign orients the standard quarter-turn frame on the orbit plane.

□

Theorem 5.6 (Rigidity of paired quarter-turn atlases)**.** *Let V be an elementary two-group and fix $\delta \neq 0$. Suppose J is an orthogonal signed-permutation operator on $\bigoplus_{\alpha \in V} \mathbb{R} e_\alpha$ of the form*

$$J(e_\alpha) = \sigma_\alpha e_{\alpha+\delta}, \qquad \sigma_\alpha \in \{\pm 1\}.$$

Then the following are equivalent:

(i) $J^2 = -I$.
(ii) $J^\dagger = -J$.
(iii) $\sigma_{\alpha+\delta} = -\sigma_\alpha$ *for every* $\alpha \in V$.

Consequently, a quarter-turn atlas with the fixed support pairing $\alpha \leftrightarrow \alpha + \delta$ is unique up to independent orientation choices on the quotient orbits $V/\langle\delta\rangle$.

Proof. On the orbit plane $\mathrm{span}\{e_\alpha, e_{\alpha+\delta}\}$, the matrix of J is

$$\begin{pmatrix} 0 & \sigma_{\alpha+\delta} \\ \sigma_\alpha & 0 \end{pmatrix}.$$

Its square is $\sigma_\alpha \sigma_{\alpha+\delta} I_2$, and its transpose equals its negative exactly when $\sigma_{\alpha+\delta} = -\sigma_\alpha$. Since the signs are ± 1, these conditions are equivalent. Choosing σ_α for one representative of each orbit determines the opposite sign on its partner. These choices are exactly the independent orientation choices on the orbit planes.

□

Remark 5.7 (What is rigid). The theorem does not say that every atlas has the same literal signs. It says that once the support pairing is fixed, the only freedom is the expected orientation reversal on each two-plane. There is no further hidden algebraic parameter in the quarter-turn itself.

6. Clifford-like sectors

The signed-XOR theorem applies immediately to Clifford basis systems [6, 10, 12]. It also explains why the result is not limited to Cayley–Dickson doubling.

Definition 6.1 (Sorted Clifford sector)**.** Let $(I, <)$ be a linearly ordered index set. The sorted Clifford sector $\mathrm{Cl}_{0,I}^{\mathrm{alg}}$ is the real associative algebra generated by $\{e_i : i \in I\}$ with relations

$$e_i^2 = -1, \qquad e_i e_j = -e_j e_i \quad (i \neq j),$$

and finite-support words only. If $J = \{j_1 < \cdots < j_m\} \in \mathrm{Fin}(I)$, write

$$e_J = e_{j_1} \cdots e_{j_m}, \qquad e_\varnothing = 1.$$

Proposition 6.2 (Sorted XOR product)**.** *For $J, K \in \mathrm{Fin}(I)$,*

$$e_J e_K = (-1)^{\mathrm{inv}(J,K)+|J\cap K|} e_{J\triangle K},$$

where

$$\mathrm{inv}(J,K) = \#\{(j,k) \in J \times K : j > k\}.$$

Proof. In the concatenated word $e_J e_K$, each inversion (j,k) with $j > k$ contributes one swap and hence one minus sign, following the standard ordered-monomial calculus for Clifford algebras [10, 12]. Each common index then contributes $e_i^2 = -1$. The remaining support is the symmetric difference $J\triangle K$.

□

Lemma 6.3 (Squares of Clifford basis monomials)**.** *If $|J| = m$, then*

$$e_J^2 = (-1)^{m(m+1)/2}.$$

In particular, $e_J^2 = -1$ exactly when $m \equiv 1$ or $2 \pmod 4$.

Proof. To compute $e_J e_J$, the second copy of e_J must be moved through the first copy. The number of swaps needed to pair equal generators is $m(m-1)/2$, and the m resulting squares each contribute -1. Hence

$$e_J^2 = (-1)^{m(m-1)/2}(-1)^m = (-1)^{m(m+1)/2}.$$

The parity of $m(m+1)/2$ is odd exactly for $m \equiv 1, 2 \pmod 4$.

□

Theorem 6.4 (Clifford-like homogeneous ALH atlas)**.** *Let $\delta \in \mathrm{Fin}(I)$ satisfy $|\delta| \equiv 1$ or $2 \pmod 4$. Then right multiplication by e_δ in the sorted Clifford sector is an admissible quarter-turn on the whole finite-support basis space:*

$$R_\delta^2 = -I, \qquad R_\delta^\dagger = -R_\delta.$$

Consequently,

$$\mathrm{Cl}_{0,I}^{\mathrm{alg}} = \bigoplus_{[J]\in \mathrm{Fin}(I)/\langle\delta\rangle} \mathrm{span}_{\mathbb{R}}\{e_J, e_{J\triangle\delta}\}$$

as a homogeneous ALH quarter-turn atlas.

Proof. The algebra is associative. Therefore, for every basis monomial e_J,

$$R_\delta^2(e_J) = (e_J e_\delta)e_\delta = e_J(e_\delta^2) = -e_J$$

by Lemma 6.3. Since right multiplication by a basis monomial is a signed permutation of the orthonormal basis, the same two-plane matrix argument

as in Theorem 4.6 gives $R_\delta^\dagger = -R_\delta$. The atlas decomposition is Theorem 5.2 with $V = \mathrm{Fin}(I)$.

□

Corollary 6.5 (Generator directions)**.** *For every generator e_i in a sorted Clifford sector,*

$$R_i(x) = xe_i$$

defines a homogeneous ALH quarter-turn atlas. The support pairing is

$$J \longleftrightarrow J\triangle\{i\}.$$

Proof. Here $|\{i\}| = 1$, so Lemma 6.3 gives $e_i^2 = -1$. Apply Theorem 6.4.

□

Definition 6.6 (Clifford-like ALH signed-XOR algebra)**.** A signed-XOR algebra $\mathbb{R}_\theta[V]$ is called *Clifford-like ALH* along a set $D \subseteq V$ if every $\delta \in D$ is right-ALH-admissible in the sense of Definition 4.5. The associated atlas family is

$$\{(R_\delta, V/\langle\delta\rangle) : \delta \in D\}.$$

Corollary 6.7 (Atlas family for Clifford-like algebras)**.** *Every Clifford-like ALH signed-XOR algebra carries a homogeneous quarter-turn atlas for each direction $\delta \in D$. The atlas is determined at support level by the quotient $V/\langle\delta\rangle$ and at orientation level by the paired signs $\theta(\alpha, \delta)$.*

Proof. This is Theorem 5.2 applied independently to each $\delta \in D$.

□

7. Cayley–Dickson layer-flow theorem

Cayley–Dickson algebras provide the tower where the dimension-flow interpretation is most explicit [2, 5, 14].

Convention 7.1 (Cayley–Dickson convention)**.** Let A_n be a real Cayley–Dickson algebra with conjugation $a \mapsto a^*$. We use the doubling

$$A_{n+1} = A_n \oplus A_n\varepsilon_n$$

with pair notation $(a, b) = a + b\varepsilon_n$ and multiplication

$$(a, b)(c, d) = \big(ac - d^*b,\ da + bc^*\big). \tag{7.1}$$

Conjugation is

$$(a, b)^* = (a^*, -b).$$

The new basis unit is

$$\varepsilon_n = (0, 1).$$

The Euclidean inner product is defined recursively by

$$\langle (a,b),(c,d)\rangle = \langle a,c\rangle_{A_n} + \langle b,d\rangle_{A_n}.$$

Lemma 7.2 (The new unit)**.** *In the convention above,*

$$\varepsilon_n^2 = -1, \qquad \varepsilon_n^* = -\varepsilon_n.$$

Moreover, right multiplication by ε_n is

$$J_n(a,b) := (a,b)\varepsilon_n = (-b,a). \tag{7.2}$$

Proof. Using (7.1),

$$(0,1)(0,1) = (-1,0) = -1.$$

Also $(0,1)^* = (0,-1) = -\varepsilon_n$. Finally,

$$(a,b)(0,1) = (-b,a),$$

again by (7.1).

□

Theorem 7.3 (Cayley–Dickson layerwise ALH flow)**.** *For every Cayley–Dickson doubling*

$$A_{n+1} = A_n \oplus A_n\varepsilon_n,$$

right multiplication by the newly adjoined unit,

$$J_n = R_{\varepsilon_n},$$

is an admissible quarter-turn on the real inner-product space A_{n+1}:

$$J_n^2 = -I_{A_{n+1}}, \qquad J_n^\dagger = -J_n.$$

It exchanges the lower layer and the companion layer by

$$J_n(A_n) = A_n\varepsilon_n, \qquad J_n(A_n\varepsilon_n) = -A_n.$$

Hence

$$J_n^{-1} = -J_n.$$

Proof. By Lemma 7.2,

$$J_n(a,b) = (-b,a).$$

Therefore

$$J_n^2(a,b) = J_n(-b,a) = (-a,-b) = -(a,b),$$

so $J_n^2 = -I$. For the adjoint,

$$\begin{aligned}\langle J_n(a,b),(c,d)\rangle &= \langle(-b,a),(c,d)\rangle \\ &= -\langle b,c\rangle + \langle a,d\rangle ,\end{aligned}$$

whereas

$$\begin{aligned}\langle (a,b), J_n(c,d)\rangle &= \langle (a,b),(-d,c)\rangle \\ &= -\langle a,d\rangle + \langle b,c\rangle .\end{aligned}$$

Thus

$$\langle J_n(a,b),(c,d)\rangle = -\langle (a,b), J_n(c,d)\rangle ,$$

which is $J_n^\dagger = -J_n$. Finally,

$$J_n(a,0) = (0,a), \qquad J_n(0,a) = (-a,0),$$

which gives the two layer identities. The inverse formula follows from $J_n^2 = -I$.

□

Corollary 7.4 (Homogeneous two-plane decomposition)**.** *Let*

$$\{e_0 = 1, e_1, \ldots, e_{2^n-1}\}$$

be the standard basis of A_n. In the doubled algebra A_{n+1} write

$$e_{r+2^n} = e_r\varepsilon_n \qquad (0 \le r < 2^n).$$

Then

$$A_{n+1} = \bigoplus_{r=0}^{2^n-1} \operatorname{span}_{\mathbb{R}}\{e_r, e_{r+2^n}\},$$

and on each summand

$$J_n(e_r) = e_{r+2^n}, \qquad J_n(e_{r+2^n}) = -e_r.$$

Each summand is an ALH quarter-turn plane.

Proof. The basis of $A_{n+1} = A_n \oplus A_n\varepsilon_n$ is the disjoint union of the lower basis $e_r = (e_r, 0)$ and its companion copy $e_{r+2^n} = (0, e_r)$. Lemma 7.2 gives the two displayed formulas. Each two-plane carries the standard quarter-turn matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

in the ordered basis (e_r, e_{r+2^n}).

□

Corollary 7.5 (No dependence on global Hurwitz properties)**.** *Theorem 7.3 remains valid at every Cayley–Dickson stage, including stages where the algebra is nonassociative, nonalternative, has zero divisors, or lacks a multiplicative quadratic norm [13, 14].*

Proof. The proof of Theorem 7.3 uses only the pair formula $J_n(a,b) = (-b,a)$ and the Euclidean inner product. It does not use associativity, alternativity, division, or norm multiplicativity.

□

8. Quaternions, octonions, sedenions, and beyond

We spell out the first stages because they make the dimension-flow interpretation transparent.

Example 8.1 (Quaternionic flow)**.** Write [4, 9]

$$\mathbb{H} = \mathbb{C} \oplus \mathbb{C}j, \qquad k = ij.$$

The layer-flow operator is

$$J_1(x) = xj.$$

On the old complex slice $\operatorname{span}_{\mathbb{R}}\{1, i\}$,

$$J_1(1) = j, \qquad J_1(i) = ij = k.$$

On the companion layer,

$$J_1(j) = j^2 = -1, \qquad J_1(k) = kj = -i.$$

Thus

$$\mathbb{H} = \operatorname{span}_{\mathbb{R}}\{1, j\} \oplus \operatorname{span}_{\mathbb{R}}\{i, k\},$$

and each summand is a quarter-turn plane for J_1. The inverse transition is $-J_1$:

$$j \xmapsto{-J_1} 1, \qquad k \xmapsto{-J_1} i.$$

This is the rigorous form of the rule that raising the old axes by the transverse imaginary dimension sends 1 to j and i to k.

Example 8.2 (Octonionic flow)**.** Write [1, 2, 14]

$$\mathbb{O} = \mathbb{H} \oplus \mathbb{H}\ell.$$

The new unit is ℓ, and

$$J_2(x) = x\ell.$$

For the quaternionic basis $1, i, j, k$,

$$1 \mapsto \ell, \qquad i \mapsto i\ell, \qquad j \mapsto j\ell, \qquad k \mapsto k\ell.$$

A second positive quarter-turn returns to the negative lower layer:

$$\ell \mapsto -1, \qquad i\ell \mapsto -i, \qquad j\ell \mapsto -j, \qquad k\ell \mapsto -k.$$

Thus

$$\mathbb{O} = \mathrm{span}_{\mathbb{R}}\{1, \ell\} \oplus \mathrm{span}_{\mathbb{R}}\{i, i\ell\} \oplus \mathrm{span}_{\mathbb{R}}\{j, j\ell\} \oplus \mathrm{span}_{\mathbb{R}}\{k, k\ell\}.$$

This decomposition is independent of octonionic nonassociativity; it is a layerwise linear quarter-turn decomposition.

Example 8.3 (Sedenionic flow)**.** Write

$$\mathbb{S} = \mathbb{O} \oplus \mathbb{O}s.$$

Let $e_0 = 1, e_1, \ldots, e_7$ be the octonion basis and set

$$e_{r+8} = e_r s \qquad (0 \le r \le 7).$$

Then

$$J_3(e_r) = e_{r+8}, \qquad J_3(e_{r+8}) = -e_r.$$

Therefore

$$\mathbb{S} = \bigoplus_{r=0}^{7} \mathrm{span}_{\mathbb{R}}\{e_r, e_{r+8}\}.$$

Although the sedenions have zero divisors and no longer have the classical Hurwitz division behavior [13], the homogeneous ALH quarter-turn atlas remains intact.

Corollary 8.4 (All higher Cayley–Dickson stages)**.** *For every $n \ge 0$, the Cayley–Dickson algebra A_{n+1} admits the homogeneous layer atlas*

$$A_{n+1} = \bigoplus_{r=0}^{2^n-1} \mathrm{span}_{\mathbb{R}}\{e_r, e_{r+2^n}\},$$

with

$$e_r \overset{J_n}{\longmapsto} e_{r+2^n} \overset{J_n}{\longmapsto} -e_r.$$

The downward transition is

$$e_{r+2^n} \overset{-J_n}{\longmapsto} e_r.$$

Proof. This is Corollary 7.4 for arbitrary n.

□

9. Relation between the signed-XOR and Cayley–Dickson pictures

The Cayley–Dickson basis can be encoded by signed-XOR support together with a recursively defined twist sign [8]. The exact sign convention depends on the Cayley–Dickson convention, but the structural separation is invariant: support toggles are XOR, while signs record orientation and associator data.

Proposition 9.1 (Cayley–Dickson support toggle)**.** *Let $A_{n+1} = A_n \oplus A_n\varepsilon_n$ and label the standard basis of A_{n+1} by $\mathbb{F}_2^{n+1}$. Let δ_n denote the new coordinate direction corresponding to ε_n. Then right multiplication by ε_n has support action*

$$\alpha \longmapsto \alpha + \delta_n.$$

Moreover, with the orientation convention of Corollary 7.4, the paired signs satisfy

$$\theta(\alpha, \delta_n)\theta(\alpha + \delta_n, \delta_n) = -1.$$

Proof. The basis element e_α in the lower copy is sent to its companion basis element $e_{\alpha+\delta_n}$, and the companion basis element is sent to $-e_\alpha$. Hence the support changes by adding δ_n in both cases, and the product of the two paired signs is -1.

□

Theorem 9.2 (Support-toggle universality of ALH layer flow)**.** *The Cayley–Dickson layer atlas is a special case of the signed-XOR homogeneous ALH atlas for the admissible direction δ_n. Conversely, every right-ALH-admissible direction δ in a signed-XOR algebra has the same support-toggle form as a Cayley–Dickson layer transition: it toggles basis labels by $\alpha \mapsto \alpha + \delta$ and decomposes the space into two-point support orbits.*

Proof. The first statement follows from Proposition 9.1 and Theorem 5.2. The converse is exactly Theorem 5.2: admissibility gives a quarter-turn on each orbit of the support toggle by δ.

□

Theorem 9.3 (Unified Hurwitz–Clifford–Cayley–Dickson principle)**.** *The local Hurwitz slice theorem, the signed-XOR atlas theorem, the Clifford-like atlas theorem, and the Cayley–Dickson layer-flow theorem are manifestations of one ALH quarter-turn mechanism.*

More precisely, the following implications hold.

(i) Every ALH generator e with $e^2 = -1$ and $e^* = -e$ determines a local complex slice

$$\mathbb{R}\,1 \oplus \mathbb{R}e \cong \mathbb{C}.$$

(ii) *Every right-ALH-admissible direction δ in a signed-XOR algebra globalizes the local slice mechanism into the homogeneous support atlas*

$$\mathbb{R}_\theta[V] = \bigoplus_{[\alpha]\in V/\langle\delta\rangle} \operatorname{span}_{\mathbb{R}}\{e_\alpha, e_{\alpha+\delta}\}.$$

(iii) *Every sorted Clifford sector $\mathrm{Cl}^{\mathrm{alg}}_{0,I}$ is an associative signed-XOR realization. If $\delta \in \mathrm{Fin}(I)$ satisfies $|\delta| \equiv 1$ or $2 \pmod 4$, then right multiplication by e_δ is an admissible ALH quarter-turn and gives the atlas*

$$\mathrm{Cl}^{\mathrm{alg}}_{0,I} = \bigoplus_{[J]\in \mathrm{Fin}(I)/\langle\delta\rangle} \operatorname{span}_{\mathbb{R}}\{e_J, e_{J\triangle\delta}\}.$$

(iv) *Every Cayley–Dickson doubling $A_{n+1} = A_n \oplus A_n\varepsilon_n$ is a distinguished signed-XOR layer-flow realization, with admissible direction given by the newly adjoined unit ε_n and with*

$$J_n = R_{\varepsilon_n}, \qquad J_n(a,b) = (-b,a), \qquad J_n^2 = -I, \qquad J_n^\dagger = -J_n.$$

Consequently

$$A_{n+1} = \bigoplus_{r=0}^{2^n-1} \operatorname{span}_{\mathbb{R}}\{e_r, e_{r+2^n}\}, \qquad e_r \mapsto e_{r+2^n} \mapsto -e_r.$$

Thus Hurwitz local complex slices, Clifford-like support geometry, and Cayley–Dickson layer transitions are not separate phenomena. They are the local, associative signed-XOR, and recursive layer-flow realizations of the same admissible ALH quarter-turn principle.

Proof. We assemble the previously proved pieces.

For (1), Theorem 2.3 proves that any element e satisfying $e^2 = -1$ and $e^* = -e$ generates a two-dimensional real $*$-algebraic slice $\mathbb{R}\,1 \oplus \mathbb{R}e$ isomorphic to $\mathbb{C}$.

For (2), Theorem 4.6 shows that the paired sign condition

$$\theta(\alpha,\delta)\theta(\alpha+\delta,\delta) = -1$$

is equivalent to $R_\delta^2 = -I$ and $R_\delta^\dagger = -R_\delta$. Theorem 5.2 then decomposes the basis into the two-point orbits of the support toggle $\alpha \mapsto \alpha + \delta$ and gives the displayed homogeneous atlas.

For (3), Proposition 6.2 identifies the sorted Clifford product with a signed-XOR product on $\mathrm{Fin}(I)$. Lemma 6.3 shows that $e_\delta^2 = -1$ exactly when $|\delta| \equiv 1$ or $2 \pmod 4$. Since the Clifford sector is associative, right multiplication by e_δ squares to $-I$ on every basis monomial. Theorem 6.4 therefore gives the displayed Clifford atlas.

For (4), Lemma 7.2 gives $\varepsilon_n^2 = -1$, $\varepsilon_n^* = -\varepsilon_n$, and the explicit right multiplication formula $J_n(a,b) = (-b,a)$. Theorem 7.3 proves $J_n^2 = -I$ and $J_n^\dagger = -J_n$, while Corollary 7.4 gives the displayed layer decomposition. Proposition 9.1 identifies the support action as addition by the new coordinate direction δ_n, and Theorem 9.2 identifies this Cayley–Dickson layer atlas as a special case of the signed-XOR homogeneous ALH atlas.

All four constructions are therefore controlled by the same two operator equations $J^2 = -I$ and $J^\dagger = -J$, together with the same support-pairing mechanism in the global finite-support cases. This proves the unified Hurwitz–Clifford–Cayley–Dickson principle. □

Theorem 9.4 (Classification of support-toggle ALH quarter-turn atlases)**.** *Let A be a finite-support real basis system with an orthonormal basis $\{e_\alpha : \alpha \in V\}$ indexed by an elementary abelian two-group V. Suppose that J is a homogeneous ALH quarter-turn operator of support-toggle type: there is a nonzero $\delta \in V$ and signs $\sigma_\alpha \in \{\pm 1\}$ such that*

$$J(e_\alpha) = \sigma_\alpha e_{\alpha+\delta} \qquad (\alpha \in V).$$

Then the following are equivalent:

(i) $J^2 = -I$.
(ii) $J^\dagger = -J$.
(iii) $\sigma_{\alpha+\delta} = -\sigma_\alpha$ *for every* $\alpha \in V$.

When these equivalent conditions hold, J determines a homogeneous ALH atlas

$$A = \bigoplus_{[\alpha] \in V/\langle\delta\rangle} \operatorname{span}_{\mathbb{R}}\{e_\alpha, e_{\alpha+\delta}\},$$

and this atlas is classified, up to independent orientation reversal on each orbit plane, by the quotient support pairing $V/\langle\delta\rangle$.

Moreover, the three classical structures considered in this paper are precisely the following distinguished realizations of this classified support-pairing mechanism:

(i) the local Hurwitz realization, obtained by restricting to a single orbit plane and applying the local slice theorem;
(ii) the Clifford realization, obtained when the signed-XOR multiplication is the sorted Clifford product on $\mathrm{Fin}(I)$ *and δ has $|\delta| \equiv 1$ or $2 \pmod 4$;*
(iii) the Cayley–Dickson realization, obtained when $V = \mathbb{F}_2^{n+1}$ and δ is the newly adjoined doubling direction.

Thus Hurwitz local slices, Clifford sectors, and Cayley–Dickson layer systems are not merely parallel examples. They are distinguished realizations of the same classified homogeneous ALH support-toggle atlas. This classification concerns support-toggle quarter-turn operators and does not classify arbitrary ALH structures.

Proof. The equivalence of (1), (2), and (3) is exactly Theorem 5.6. Indeed, on each two-dimensional orbit plane $\operatorname{span}_{\mathbb{R}}\{e_\alpha, e_{\alpha+\delta}\}$, the matrix of J has the form

$$\begin{pmatrix} 0 & \sigma_{\alpha+\delta} \\ \sigma_\alpha & 0 \end{pmatrix},$$

so $J^2 = -I$ and $J^\dagger = -J$ are both equivalent to $\sigma_{\alpha+\delta} = -\sigma_\alpha$.

The atlas decomposition follows by decomposing V into the two-point orbits of the support toggle $\alpha \mapsto \alpha + \delta$. The only remaining freedom

is the choice of one sign on each orbit, and changing that sign reverses the orientation of the corresponding two-plane. Hence the support-level classification is by $V/\langle\delta\rangle$, with independent orientation choices on the orbit planes.

For the local Hurwitz realization, restrict the construction to one orbit plane and apply Theorem 3.2 or, equivalently, Theorem 2.3. This gives the local complex slice.

For the Clifford realization, Proposition 6.2 identifies the sorted Clifford product with a signed-XOR product on $\mathrm{Fin}(I)$, and Lemma 6.3 identifies the directions with $e_\delta^2 = -1$. Theorem 6.4 then shows that precisely such Clifford directions realize the classified ALH support-toggle atlas.

For the Cayley–Dickson realization, Proposition 9.1 identifies right multiplication by the new unit with addition by the new coordinate direction, while Theorem 7.3 and Corollary 7.4 give the quarter-turn equations and the layer decomposition. Theorem 9.2 identifies this construction as the Cayley–Dickson instance of the signed-XOR homogeneous atlas.

Therefore the local Hurwitz, Clifford, and Cayley–Dickson cases are all obtained by imposing different global algebraic realizations on the same classified support-pairing quarter-turn mechanism. □

10. Functoriality and reindexing

The support-level construction is stable under relabeling.

Definition 10.1 (Atlas-preserving map)**.** Let (V,δ) and (W,η) be elementary two-groups with distinguished nonzero directions. A group homomorphism

$$f : V \to W$$

is *direction-preserving* if $f(\delta) = \eta$. It induces a map

$$\bar{f} : V/\langle\delta\rangle \to W/\langle\eta\rangle, \qquad [\alpha] \mapsto [f(\alpha)].$$

Proposition 10.2 (Support atlas functoriality)**.** *A direction-preserving homomorphism $f : (V,\delta) \to (W,\eta)$ sends support-toggle orbits for δ to support-toggle orbits for η:*

$$f(\alpha+\delta) = f(\alpha) + \eta.$$

If f is injective, then the induced map on orbit sets is injective.

Proof. The displayed identity follows from additivity and $f(\delta) = \eta$. If f is injective and $[f(\alpha)] = [f(\beta)]$, then $f(\alpha)+f(\beta)$ is either 0 or $\eta = f(\delta)$. Hence $\alpha+\beta$ is either 0 or δ, so $[\alpha] = [\beta]$.

□

Corollary 10.3 (Finite-support naturality)**.** *Let $I \hookrightarrow I'$ be an injection of index sets and let $\delta \in \mathrm{Fin}(I)$. The induced injection*

$$\mathrm{Fin}(I) \hookrightarrow \mathrm{Fin}(I')$$

preserves the homogeneous ALH support atlas associated with δ.

Proof. The injection sends symmetric difference with δ to symmetric difference with the image of δ. Apply Proposition 10.2.

□

11. Interpretation: dimension-flow, not fractional dimension

The results above justify a precise use of dimension-flow language. They do not redefine vector-space dimension as a real, complex, or non-real scalar.

Definition 11.1 (ALH dimension-flow)**.** In a signed-XOR algebra, the *ALH dimension-flow along δ* is the operator

$$R_\delta : x \mapsto xe_\delta$$

when δ is right-ALH-admissible. In a Cayley–Dickson doubling, the *layer dimension-flow* is

$$J_n = R_{\varepsilon_n} : A_{n+1} \to A_{n+1}.$$

Proposition 11.2 (Positive and negative flow)**.** *For every ALH dimension-flow operator J constructed above,*

$$J^2 = -I, \qquad J^{-1} = -J.$$

Thus the positive flow is a $+\pi/2$ turn on each atlas plane and the inverse flow is a $-\pi/2$ turn.

Proof. This is Corollary 5.4 in the signed-XOR case and Theorem 7.3 in the Cayley–Dickson case.

□

Remark 11.3 (Terminology). The phrase "imaginary dimension-flow" refers to an orientation and layer-transition operator. It is not a claim that the cardinal dimension of a vector space is a non-real scalar. Ordinary vector-space dimension remains a natural number or a cardinal. The new datum is the quarter-turn orientation attached to a layer transition.

12. Conclusion

ALH begins with the elementary local observation that

$$e^2 = -1, \qquad e^* = -e \quad \Longrightarrow \quad \mathbb{R}\,1 \oplus \mathbb{R}e \cong \mathbb{C}.$$

From this viewpoint, the principal structural result is a unified Hurwitz–Clifford–Cayley–Dickson framework. The local Hurwitz slice, the Clifford-like atlas, and the Cayley–Dickson layer-flow construction are not independent phenomena. They arise from the same ALH quarter-turn principle expressed at different algebraic scales. The signed-XOR atlas serves as the common

support-level model, while admissible quarter-turn operators provide the common geometric mechanism.

The structural content of the present paper is that, in finite-support signed-XOR systems, such local slices assemble into homogeneous quarter-turn atlases. For a right-ALH-admissible support direction δ, right multiplication by e_δ is a skew-adjoint complex structure on the whole real basis space and decomposes that space into uniform two-dimensional ALH planes indexed by $V/\langle\delta\rangle$.

The Cayley–Dickson tower exhibits the same structure in layer form. In every doubling

$$A_{n+1} = A_n \oplus A_n\varepsilon_n,$$

the newly adjoined unit defines

$$J_n(a, b) = (-b, a),$$

so the lower layer is carried to the companion layer and the companion layer is carried to the negative lower layer. The quaternionic rule

$$1 \mapsto j, \qquad i \mapsto k$$

is therefore not an isolated accident. It is the first instance of the general layerwise quarter-turn law

$$e_r \mapsto e_{r+2^n} \mapsto -e_r,$$

valid for octonions, sedenions, and all later Cayley–Dickson stages.

This reframes the role of ALH relative to the classical Hurwitz and Cayley–Dickson literature [5, 7, 14]. It is not an escape from Hurwitz's theorem and not a claim of global division in high dimension. It is the homogeneous operator quarter-turn geometry that remains after the global Hurwitz properties have disappeared. At the support level, the phenomenon is universal for admissible support-toggle directions; at the orientation level, the twist signs choose the local frames; at the associator level, the failure of associativity is recorded separately. Thus ALH supplies a stable local and layerwise calculus across Clifford-like algebras, Cayley–Dickson algebras, and arbitrary finite-support index systems.

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